Euclidean Saddles, Calculable Models, & Bounded Topologies: A Status Report

Observers, Wormholes & Complex Saddles
Lausanne, 18th May, 2026

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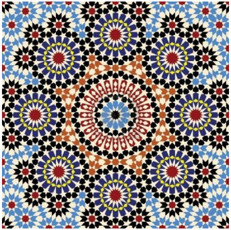
[2009.12464,2505.11330,2510.19652,2512.16738,2602.19812] & collaborations with Tarek Anous, Chiara Baracco, Pietro Benetti Genolini, Sam Brian, Frederik Denef, Damián Galante, Silvia Georgescu, Ellie Harris, Thomas Hertog, Diego Hofman, Joel Karlsson, Y. T. Albert Law, Vasileios Letsios, Chawakorn Maneerat, Ruben Monten, Beatrix Mühlmann, Guillermo Silva, Andy Svesko, Zimo Sun

Preamble



What is Cosmology?

Our current understanding of cosmology needs to be supplemented by concrete calculable/observable quantities. In the absence of these, it is difficult to begin addressing the question: What is cosmology? in the same way we wish to address: What is quantum field theory/a black hole/string theory?

Cosmological Signatures

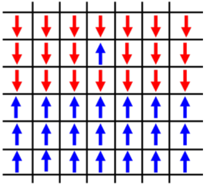
Lorentzian: One idea is to construct relational quantities, where we measure a physical phenomenon against some other semiclassical feature, e.g. the value of the inflaton field, or the average temperature of the Universe, or the time registered along an inertial observer's worldline. In these cases one must grapple with semiclassics and model dependence.

vs.

Euclidean: Here there are concrete calculables in the theory, like the partition function. Though less model dependent, we must grapple with their physical interpretation, and to do so, perhaps ultimately resort to relational notions also.

Cosmological 'Ising' Models

Even if we agree on what we should calculate, we must also identify tractable theoretical models to test our understanding with precision. This is in the same spirit as 't Hooft's solution of two-dimensional large-N QCD, or the SYK model for black holes. This may require us to depart from four-dimensions and forego (for now) phenomenological constraints.



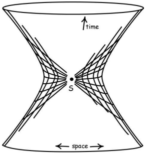
[since 2017: D.A.,Galante,Hofman-Cotler,Jensen,Maloney-Maldacena,Turiaci,Yang-Coleman,Mazenc,Shyam,Silverstein,Soni,Torroba,Yang-Narovlansky,Verlinde-Collier,Eberhardt,Mühlmann-Susskind-Blommaert,Mertens,Papalini-Betzios-many more]

An Idealised Cosmology

We will take the view of an idealised the de Sitter universe

$$\frac{ds^2}{\ell^2} = -dT^2 + \cosh^2 T d\Omega^2 \qquad \text{with} \qquad \Lambda \equiv +\frac{3}{\ell^2}$$

which exhibits maximal $SO(1,4)$ isometries. It is a type of dynamical attractor, exhibiting no notion of spatial/null infinity to build observables. We can contrast this to the ∞-number of physical couplings in EFT.

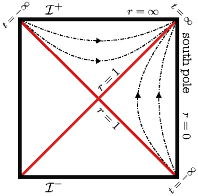


Cosmological Event Horizon

Due to the expansion, the spacetime develops a cosmological horizon, retaining a manifest symmetry $SO(1,1) \times SO(3) \subset SO(4,1)$. The static patch metric is

$$\frac{ds^2}{\ell^2} = -\sin^2\psi dt^2 + d\psi^2 + \cos^2\psi d\Omega_2^2$$

The horizon at $\psi = 0$ exhibits a Hawking temperature $\beta_{\mathrm{dS}} = \frac{2\pi\ell}{\hbar c}$. The cosmological horizon renders observables even more subtle.



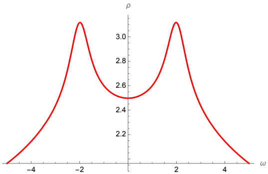
[Gibbons,Hawking-Figari,Hoegh,Krohn,Nappi-...]

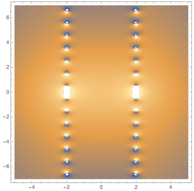
Group Theoretic Building Blocks

Unitary irreps are denoted by (Δ, s). For $s = 0$, we have $\Delta = \frac{3}{2} + i\nu$, $\Delta \in (0, 3)$, and $\Delta \in \mathbb{Z}^+$. The principal series irrep has $\Delta = \frac{3}{2} + i\nu$ and

$$\chi_{\Delta,s}(t) = \text{tr}_{\Delta,s} e^{-iHt} = (2s+1)\,\frac{e^{-\Delta t} + e^{-(3-\Delta)t}}{|1 - e^{-t}|^3} = \int_{\mathbb{R}} d\omega \rho_{\Delta,s}(\omega) e^{-i\omega t}$$

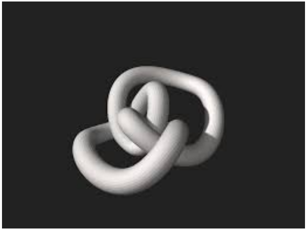
Tensor products are broadly, but not completely, understood. Interestingly, they contain (almost) all UIRs, e.g. $\mathcal{H}_{\text{gravitino}} \subset \mathcal{H}_{\text{electron}} \otimes \mathcal{H}_{\text{photon}}$.





[HarishChandra-...-Hirai-...-Basile,Bekaert,Boulanger-...]

Euclidean Saddles

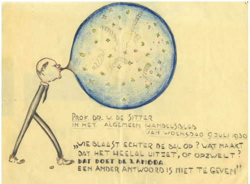


Euclidean De Sitter

The Euclidean de Sitter geometry is the round metric on the four-sphere

$$\frac{ds^2}{\ell^2} = \sin^2\psi d\xi^2 + d\psi^2 + \cos^2\psi d\Omega_2$$

Here, $\xi \sim \xi + 2\pi$. The Euclidean isometry group is $SO(5)$. The continuation $\xi \to it$ maps S^4 to the static patch. We are led to the puzzling question: What is Euclidean quantum gravity on a closed manifold?

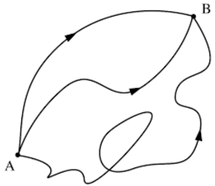


$\Lambda > 0$ Partition Function

Concretely, Euclidean quantum gravity instructs us to compute

$$\exp \mathcal{S} \equiv \sum_{\text{closed } \mathcal{M}} \int [\mathcal{D}g_{ij}]\, e^{-S_E[g_{ij},\Lambda]} \mathcal{Z}_{\text{matter}}[g_{ij}; \mathcal{M}]$$

$\mathcal{S}$ is a diff. & local field redefinition invariant calculable. The dominant Euclidean saddle for $\Lambda > 0$ is the round sphere, but there are other saddles as well. One must make sense of the conformal mode problem and a sensible sum over closed four-manifolds. We will take a saddle point perspective.



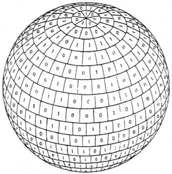
[Gibbons,Hawking-Chritensen,Duff-Fradkin,Tesytlin-Yasuda-...]

A Physical Interpretation?

$\mathcal{S}$ is rather enigmatic, and mirrors many of the Lorentzian puzzles (Hilbert space, rigid asymptotia, unitarity, etc). To leading order

$$\mathcal{S} \approx \mathcal{S}_0 \equiv \frac{3\pi c^3}{\Lambda G \hbar}$$

The quantity $\mathcal{S}_0$ is the area of the de Sitter horizon divided by $4G$. It has been speculated to encode a quantum entropy/information content. In our world $\mathcal{S}_0 \approx 10^{123}$, reminiscent of the cosmological constant problem.



[Gibbons,Hawking-...-Banks-Fischler-Parikh,Verlinde-...]

Non-Spherical Saddles

More generally, we can consider smooth Einstein metrics with $R = 4\Lambda$.

$$S^4, \quad \mathbb{C}P^2, \quad S^2 \times S^2, \quad \mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}\ , \qquad k = 1, 2, \ldots, 8$$

The volumes of all these manifolds are known. One finds

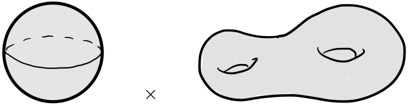
$$\mathcal{S}_{S^4} > \mathcal{S}_{\mathbb{C}P^2} > \mathcal{S}_{S^2 \times S^2} > \mathcal{S}_{\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}} > \mathcal{S}_{\mathbb{C}P^2 \# 2\overline{\mathbb{C}P^2}} > \ldots > \mathcal{S}_{\mathbb{C}P^2 \# 8\overline{\mathbb{C}P^2}}$$

Non-spherical saddles allow for additional invariant data, and may lead to violations of the de Sitter symmetries. S^4 and $S^2 \times S^2$ admit a spin-structure, the rest admit only a generalised version and are only admissible for particular spectra. The Standard Model can only be placed on these if we gauge $U(1)_{B-L}$.

[Page-Tian,Yau-Chen,LeBrun,Weber-...-Garcia-Etxebarria,Montero-...]

Einstein-Maxwell with $\Lambda > 0$

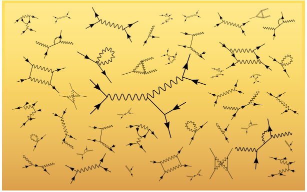
Einstein-Maxwell-Λ theory admits further saddles/topologies. Some of these saddles can be viewed as Euclidean versions of rotating, charged Nariai black holes. One can also have saddles such as $S^2 \times \Sigma_g$.



Upon cutting along an S^1, these contributions may lead to violations of the Wheeler-DeWitt equation.

[e.g. Brill-Flaherty-Gibbons,Pope-...-LeBrun-...-D.A.,Brian-]

Quantum Contributions



One-loop Structure

For general matter fields at one-loop we have the general structure

[D.A.,Denef,Law,Sun]

$$\log \mathcal{Z}_{\text{matter}} = \log \frac{1}{\text{vol}\,\mathcal{G}_q} + \int_{\varepsilon} \frac{d\mathfrak{t}}{2\mathfrak{t}} \frac{1+e^{-\mathfrak{t}}}{1-e^{-\mathfrak{t}}}\, \tilde{\chi}_{\Delta,s}(\mathfrak{t})$$

Here, $\tilde{\chi}_{\Delta,s} \equiv \chi_{\Delta,s} - \alpha_s\, \chi^{(\text{e})}_{\Delta,s}$, with $\alpha_s = \frac{1}{6}s(s+1)(2s+1)$. The contribution from $\chi^{(\text{e})}_{\Delta,s}$ might be explained from horizon edge modes. Heat kernel coefficients are coded in the small-$\mathfrak{t}$ regime, predicting bulk-edge splits.

[Buidovich,Polikarpov-Kabat-Donnelly,Wall-Dowker-Casini,Huerta-Soni,Trivedi-David,Mukherjee-Law-...]

Precision Data

The one-loop pre-factors of S^4, $\mathbb{C}P^2$, and $S^2 \times S^2$ are calculable. For $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ one can has numerical spectra. The phase $(\pm i)^{\mathcal{P}}$ is also computable, e.g. $\mathcal{P}_{S^4} = 6$, $\mathcal{P}_{\mathbb{C}P^2} = 1$, $\mathcal{P}_{S^2 \times S^2} = \mathcal{P}_{\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}} = 0$.

$$\log |\mathcal{Z}^{(1)}_{S^4}| = \frac{24}{\varepsilon^4 \Lambda^2} - \frac{32}{\varepsilon^2 \Lambda} - \frac{571}{45} \log\left(\sqrt{\frac{3}{\Lambda}} \frac{2}{\varepsilon\, e^{\gamma}}\right) + \frac{715}{48} - \log 2 - \frac{47}{3}\zeta'(-1) + \frac{2}{3}\zeta'(-3) - 5\log\frac{\mathcal{S}_{S^4}}{2\pi} - \log \mathrm{vol}_{\mathrm{SO}(5)}$$

$$\log |\mathcal{Z}^{(1)}_{\mathbb{C}P^2}| = \frac{18}{\varepsilon^4 \Lambda^2} - \frac{24}{\varepsilon^2 \Lambda} - \frac{359}{60} \log\left(\sqrt{\frac{3}{\Lambda}} \frac{2}{\varepsilon\, e^{\gamma}}\right) + \frac{25}{3} - \frac{1081}{120}\log 2 + \frac{7}{2}\log 3 - 12\zeta'(-1) + 2\zeta'(-3) - 4\log\frac{\mathcal{S}_{\mathbb{C}P^2}}{2\pi} - \log \mathrm{vol}_{\mathrm{SU}(3)}$$

$$\log |\mathcal{Z}^{(1)}_{S^2 \times S^2}| = \frac{16}{\varepsilon^4 \Lambda^2} - \frac{64}{3\varepsilon^2 \Lambda} - \frac{98}{45} \log\left(\sqrt{\frac{3}{\Lambda}} \frac{2}{\varepsilon\, e^{\gamma}}\right) + \gamma_0 - 3\log\frac{\mathcal{S}_{S^2 \times S^2}}{2\pi} - \log \mathrm{vol}_{\mathrm{SO}(3)^2}$$

[Polchinski-Volkov,Wipf-D.A.,Baracco,Brian,Denef-Ivo,Maldacena,Sun-Turiaci,Wu-Law,Lochab-...]

Beyond One Loop

There is an all loop proposal for S^3 partition function

$$\exp \mathcal{S} = \pm i\, e^{\mathcal{S}_0} \left| \sqrt{\frac{4\pi}{\mathcal{S}_0}} \sinh \frac{4\pi^2}{\mathcal{S}_0} \right|^2$$

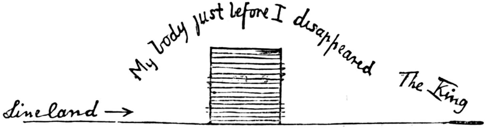
with $\mathcal{S}_0 = \frac{\pi}{2G\sqrt{\Lambda}} \in \mathbb{R}^+$. In the semiclassical limit $\mathcal{S}_0 \gg 1$, such that

$$\exp \mathcal{S} \approx \pm i\, e^{\mathcal{S}_0} \times \left(\frac{1}{\text{vol}\, SU(2)} \right)^2 \times \left(\frac{8\pi^3}{\mathcal{S}_0} \right)^3 \times \left(1 + \frac{4\pi^4}{3} \frac{1}{\mathcal{S}_0^2} + \dots \right)$$

The all-loop expression contrasts the one-loop exactness of the Euclidean AdS_3 result. It can be used to constrain microscopic proposals.

[Witten-Carlip-Guadagini, Tomassini-Castro, Lashkari, Maloney-Gukov, Mariño, Putrov-D.A., Denef, Law, Sun-Collier, Eberhardt, Mühlmann-...]

Two-Dimensional Quantum Cosmology



2D Quantum Model

Two-dimensional quantum gravity coupled to conformal matter

$$\exp \mathcal{S} = \sum_{h=0}^{\infty} e^{\vartheta \chi_h} \int [\mathcal{D} g_{ij}] e^{-\Lambda \int \sqrt{g}} \mathcal{Z}_{\mathrm{CFT}}^{(h)}[g_{ij}; c]$$

is tractable due to its relation to Liouville theory. The exact Λ-dependence is

$$\log \mathcal{Z}_{\mathrm{grav}}^{(h)} = \vartheta \chi_h - \frac{\chi_h}{2} \left(\frac{24}{\left(\sqrt{1-c} - \sqrt{25-c}\right)^2} + 1 \right) \log \frac{1}{\Lambda} + f^{(h)}(c)$$

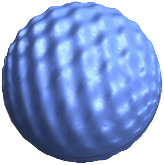
Many other quantities are also exactly calculable.

[Knizhnik,Polyakov,Zamolodchikov-Distler,Kawai-David-Dorn,Otto-Zamolodchikov^2-...]

Semiclassical Limit

In general, the metric has $\mathcal{O}(1)$ fluctuations. Nevertheless, for $c \to \infty$ one can tame the metric fluctuations, and built a perturbative expansion in $\frac{1}{c}$. In the (super)-conformal gauge one is led to a (super)-timelike Liouville theory, adding motivation to solve this theory. For example, in this regime $\mathcal{P}_{S^2} = 0$ and

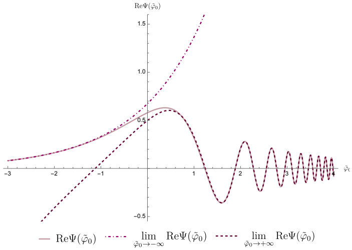
$$\log \mathcal{Z}^{(0)}_{\text{grav}} \approx 2\vartheta + \left(\frac{c}{6} - \frac{19}{6} - \frac{6}{c} + \ldots\right) \log \frac{1}{\Lambda} + f^{(0)}(c)$$



[Polchinski-...-Harlow,Maltz,Witten-...-D.A.,Bautista,Mühlmann-Mühlmann-D.A.,Genolini,Mühlmann-Chatterjee-...]

Lorentzian Picture

The classical phase space exhibits big bang/crunch geometries, and bounce solutions. The big bang singularity stems from a highly excited matter state in the Cardy regime. So there is a degeneracy of big bangs. One also uncovers exact solutions to the Wheeler-DeWitt equation. The Hartle-Hawking wavefunction, Ψ_{HH}, is coded in the timelike Liouville disk path integral.



[Carneiro da Cunha,Martinec-...-D.A.,Baracco,Mühlmann-D.A.,Hertog,Karlsson-Giribet,Sivilotti]

What is Timelike Liouville Theory?

Path integral: One way to (formally) describe timelike Liouville is through

$$S_{\mathrm{tL}} = \frac{1}{4\pi}\int d^2x\sqrt{\tilde{g}}\left(-\tilde{g}^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - q\tilde{R}\varphi + 4\pi\Lambda e^{2\beta\varphi}\right)$$

A path-integral treatment of the above action yields a non-vanishing sphere path integral, and disk amplitudes that do not obey the reflection property $\mathcal{V}_\alpha = \mathcal{V}_{-(q+\alpha)}$ of Liouville vertex operators. One needs a choice of contour.

[D.A.,Bautista,Mühlmann-Chatterjee-Usciati,Guillharmou,Rhodes,Santachiara]

vs.

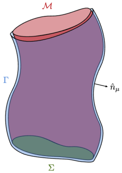
Conformal bootstrap: The proposed timelike DOZZ formula appears to predict a vanishing sphere path integral and reflection symmetric structure constants. Part of the issue is that timelike Liouville theory is non-unitary.

[Schomerus-Kostov,Petkova-Zamolodchikov-Harlow,Maltz,Witten-Ribault,Santachiara]

Quasilocal Features

Manifolds with Finite Boundaries

In Euclidean signature, Anderson has established ellipticity of conformal boundary conditions (CBC), whereby the conformal class γ_{mn} and tr$K$ are fixed. This constrasts the Dirichlet/Neumann case.

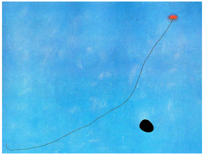


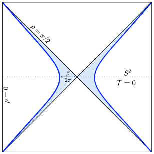
In Lorentzian signature the problem is open (physically and mathematically). Generically, Dirichlet and Neumann are not well posed due to uniqueness and existence issues. But they can display good properties. CBC has good uniqueness/existence properties. The issue of stability is unresolved.

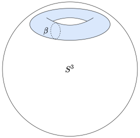
[An,Anderson-D.A.,Galante,Maneerat-Silverstein et. al.-Shaghoulian et. al.-Liu,Santos,Wiseman-...]

Worldline & Stretched Horizon

One might view these boundaries as slightly thickened worldlines that act as gravitational observatories for spacetimes without asymptotic boundaries. Alternatively, the surface can be pushed near a horizon, and play the role of a stretched horizon/membrane. In this limit, $\mathrm{tr}K\ell \to \infty$.







[Damour-Znajek-Price, Thorne-'tHooft-Susskind, Thorlacius, Uglum]

[D.A., Hofman, Hartnoll-Loganayagam, Shetye-Chandrasekaran, Longo, Penington, Witten-Maldacena-Chen, Stanford, Tang, Yang]

Stretched Horizon Dynamics

As $\mathrm{tr}K\ell \to \infty$, one uncovers

$$\begin{aligned} \omega_L \ell &= \pm\frac{1}{\sqrt{2}\mathrm{tr}K\ell}\sqrt{L(L+1)} - \frac{i}{4\mathrm{tr}K^2\ell^2}\left(L(L+1)-2\right)+\dots \\ \omega_L \ell &= -\frac{i}{2\mathrm{tr}K^2\ell^2}\left(L(L+1)-2\right)+\dots \end{aligned}$$

These are reminiscent of 'fluid-gravity' sound and shear modes of a conformal fluid with vanishing bulk viscosity and $\frac{\eta}{s}=\frac{1}{4\pi}$. In addition there are 'soft' modes $\omega_L^{(\pm)} = \pm 2\pi i T_{\mathrm{dS}}$ for each L, whose physical picture is unclear. Generally, the approach may offer a systematic toolkit for the membrane paradigm.

[Strominger,Bredberg-D.A.,Anous,Bredberg,Ng-D.A.,Galante,Georgescu,Maneerat,Svesko]

Euclidean CBC

In Euclidean signature, finite boundaries play an important role in the (Gibbons+Hartle)-Hawking path integral. Here we imagine computing

$$\mathcal{Z}[\gamma_{mn}, \mathrm{tr}K] = \int \mathcal{D}g_{\mu\nu}\, e^{-S_E[g_{\mu\nu}]}$$

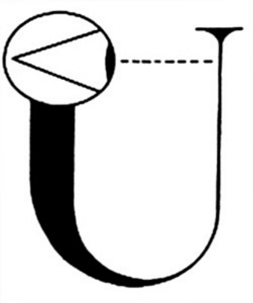
for configurations on a manifold $\mathcal{M}$ with non-vanishing boundary $\partial\mathcal{M}$. For Euclidean black holes one finds interesting high temperature behaviour, e.g. large AdS black holes:

$$S_{\mathrm{BH}} = \frac{16\pi^3}{243|\Lambda|G}\left(\mathrm{tr}K\ell - \sqrt{\mathrm{tr}K^2\ell^2 - 9}\right)^2 \frac{1}{\beta^2} + \ldots$$

Recent work proposes a holographic interpretation (3d & $\Lambda < 0$) in terms of a deformed CFT coupled to timelike Liouville theory.

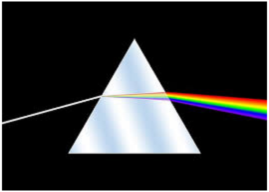
[Allameh,Shaghoulian]

Outlook



Spectroscopy

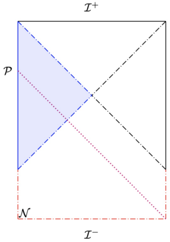
One can sum up perturbative particle spectra for candidate theories. Even at one-loop this might reveal interesting structure. For example, we might see glimmers of a genus-one worldsheet structure from target space spectra summed at one-loop. In $\mathrm{AdS}_4/\mathrm{CFT}_3$, the sum over bulk one-loop diagrams can reveal properties of the dual CFT_3.

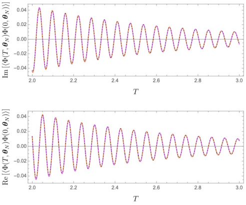


[Giombi,Klebanov-...-D.A.,Baracco,Letsios,Mühlmann]

Black Hole vs. Cosmological Horizons

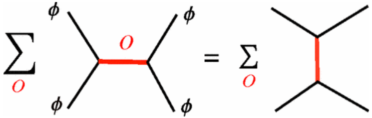
Energy pulses open up the de Sitter horizon, connecting the left and right static patches. This is a consequence of a theorem of Gao and Wald. It is reminiscent of the reentering of super-horizon modes in inflationary cosmology. Moreover, heavy particles exhibit oscillatory correlations, while light particles have very long range correlations. These effects seem distinctive to cosmological horizons, and their underlying (non)-chaotic features.





[D.A.,Hofman,Galante-D.A.,Galante,Mühlmann-Aalsma,Shiu-...-Narovlanski-]

Due to time constraints I have omitted many nice developments on quantum fields on a rigid de Sitter space. These range from conformal bootstrap ideas to the construction of solvable/integrable quantum field theories.



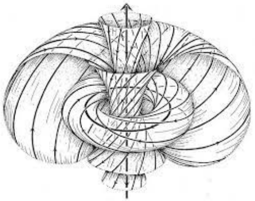
[e.g. ...-Sleight,Taronna-Hogervorst,Penedones,Vaziri-DiPietro,Gorbenko,Komatsu-ArkaniHamed,Baumann,Lee,Pimentel-Benincasa-D.A.,Anous,RiosFukelman...]

Gravitational Gluing

To connect to S^4 we are led to consider the more general path-integral

$$(\mathcal{Z}_1, \mathcal{Z}_2) \equiv \int \frac{[\mathcal{D}\gamma]}{\text{vol}\,(\text{diff} \times \text{Weil})} \mathcal{Z}_1^*[-\text{tr}K, \gamma_{ij}]\mathcal{Z}_2[\text{tr}K, \gamma_{ij}]$$

At the saddle point level in all dimensions & all loops in two-dimensions, $(\mathcal{Z}, \mathcal{Z})$ is independent of $\text{tr}K$. It is suggestive of an inner product structure for Ψ on a finite spatial slice.



[D.A.,Denef,Monten,Sun-Collier,Eberhardt,Mühlmann-D.A.,Galante,Maneerat-D.A.,Hertog,Karlsson-Cotler,Jensen-...]

Sharp, concrete calculables, and simplified models at the quantum level may bring us a step closer to understanding quantities such as $\mathcal{S}$, and the cutting and gluing rules of $\mathcal{Z}_{\text{grav}}$. I don't know how all the pieces fit together (yet). Our work is cut out for us.

Thank you for your Time!

Higher Spin Spectroscopy

Infinite tower of higher (even) spin Fronsdal gauge fields yields ($q \equiv e^{-\mathfrak{t}}$)

$$\int_{\mathbb{R}^+} \frac{d\mathfrak{t}}{2\mathfrak{t}} \frac{1+q}{1-q} \left(\chi^{(0)} + \sum_{s=1}^{\infty} \left(\chi^{(2s)}_{\text{bulk}} - \chi^{(2s)}_{\text{edge}} \right) \right)$$
$$= \int_{\mathbb{R}^+} \frac{d\mathfrak{t}}{2\mathfrak{t}} \frac{1+q}{1-q} \frac{-q}{(1-q)^2} + 2 \int_{\mathbb{R}^+} \frac{d\mathfrak{t}}{2\mathfrak{t}} \frac{1+q}{1-q} \frac{q^{\frac{1}{2}} + q^{\frac{3}{2}}}{(1-q)^2}$$

The resulting expression has no logarithmic divergence. Instead, it has a three-dimensional ultraviolet structure.

Dimensional Metamorphosis

The resulting pieces are identifiable as follows.

The 1st piece is

$$\log \mathcal{Z}_{\text{chs}} = \int_{\mathbb{R}^+} \frac{dt}{2t} \frac{1+q}{1-q} \frac{-q}{(1-q)^2}$$

the one-loop partition function of 3d conformal higher spin gauge theory on S^3.

The 2nd piece is

$$\log \mathcal{Z}_{\text{free}} = 2 \int_{\mathbb{R}^+} \frac{dt}{2t} \frac{1+q}{1-q} \frac{q^{\frac{1}{2}} + q^{\frac{3}{2}}}{(1-q)^2}$$

twice the partition function of a free conformally coupled scalar on S^3.

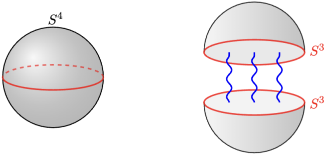
[Beccaria,Bekaert,Tseytlin-Klebanov,Pufu,Safdi]

Gluing Formula for $\mathcal{Z}_{\text{H.S.}}[S^4]$

The higher spin sums suggest an expression of the form

$$\mathcal{Z}_{\text{h.s.}}[S^4] = \frac{i^{\mathcal{P}}}{\text{vol}\,\mathcal{G}_{\text{HS}}} \int [\mathcal{D}\mathcal{B}]\, |\mathcal{Z}_N[\mathcal{B}]|^2$$

where $\mathcal{Z}_N[\mathcal{B}]$ is the partition function of a 3d Sp(N) vector model on S^3, with $N \in \mathbb{Z}^+$. The $\mathcal{B}$ are sources for all conserved currents + scalar. To leading order at large N, we have $\mathcal{Z}_{\text{h.s.}}[S^4] \approx e^{2N\left(\frac{1}{8}\log 2 - \frac{3\zeta(3)}{16\pi^2}\right)} e^{+\frac{\zeta(3)}{8\pi^2}}$. Note that $Sp(N) = O(-N)$ is crucial for positive $\mathcal{S}_0$.



[D.A.,Baracco,Letsios,Mühlmann]

An '$\mathcal{N}=2$' Extension

We can consider bulk spectra with $s = \{0, \frac{1}{2}, 1, \frac{3}{2}, 2, ...\}$. The $\mathcal{N}=2$ extension of the gluing formula becomes

$$\mathcal{Z}^{\mathcal{N}=2}_{\text{h.s.}}[S^4] = \frac{i^{\mathcal{P}_{\mathcal{N}=2}}}{\text{vol}\,\mathcal{G}_{\text{sHS}}} \int [\mathcal{D}\mathcal{B}_s] \left| \mathcal{Z}^{\mathcal{N}=2}_{N}[\mathcal{B}_s] \right|^2$$

where $\mathcal{Z}^{\mathcal{N}=2}_{N}[\mathcal{B}_s]$ is the partition function of a 3d $\mathcal{N}=2$ free $U(-N)$ vector model on S^3, and the $\mathcal{B}_s$ are sources for all conserved (super)-currents. To leading order we compute

$$\mathcal{Z}^{\mathcal{N}=2}_{\text{h.s.}}[S^4] \approx \exp\left(2N\left(\frac{\log 2}{4} - \frac{3\zeta(3)}{8\pi^2}\right)_b + 2N\left(\frac{\log 2}{4} + \frac{3\zeta(3)}{8\pi^2}\right)_f\right) = 2^N$$

$\mathcal{N} = 2$ at One-loop

Due to a series of remarkable cancelations, at one-loop all super-conformal higher spin contributions vanish. The overall phase $\mathcal{P}_{\mathcal{N}=2}$ vanishes also.

$$\mathcal{Z}_{\text{h.s.}}^{\mathcal{N}=2}[S^4] \approx 2^N \times \frac{N^{-\frac{1}{16}}}{\operatorname{vol} \mathcal{G}_{\text{sHS}}}$$

SUSY localisation might establish one-loop exactness. For $N_1 = n_1^{16} k$, $N_2 = n_2^{16} k$ with $\{k, n_1, n_2\} \in \mathbb{Z}^+$, we find the ratio

$$\mathfrak{r} = 2^{N_1 - N_2} \left(\frac{N_2}{N_1} \right)^{\frac{1}{16}} \in \mathbb{Q}$$

Outlook

More generally, we can hope for a structure

$$\mathcal{Z}_{\text{grav}} = \sum_{n \in \text{saddles}} e^{\alpha_n \mathcal{S}_0 + \gamma_n \log \mathcal{S}_0} \sum_{l \in \mathbb{N}} \beta_{nl}\, \mathcal{S}_0^{-l} \ ,$$

where $\mathcal{S}_0 \sim \frac{1}{G_N \Lambda}$. The $\{\alpha_n, \beta_{nl}\}$ are generically transcendental numbers fixed by the properties of the number of matter fields, the structure of the interactions, and so on. The γ_n are often found to be rational numbers.

It is conceivable that in a more generic theory, $G_N\Lambda$ is still a function of some integer or takes a discrete set of allowed values. If we further require that a consistent theory sets $\mathcal{Z}_{\text{grav}}$ to be integer valued, or that the ratio of two $\mathcal{Z}_{\text{grav}}$ be rational valued, or something else of the like, those values within the discretuum of $G_N\Lambda$ that are still allowed may be sporadic with no simple or uniform distribution. Mathematical equations, like the Diophantine equations, whose variables are restricted to discrete sets can display such properties.