Edge Modes on Stringy Horizons

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Based on [2601.13131] with Atish Dabholkar and Upamanyu Moitra

Overview

- **Motivation**
  - Euclidean path integrals and thermal partition functions
  - QFT in de Sitter: bulk + edge
- **QFT $\rightarrow$ String theory**
  - Warm-up: sum over bulk modes in $L \rightarrow \infty$ limit
  - Modular averaging
- **Main result: edge modes in string theory**
  - A modular invariant worldsheet partition function
- **Outlook**

Euclidean = Thermal?

[D. Anninos, F. Denef, A. Law, Z. Sun; (2020)]

For a Euclidean manifold of the form $M \times S^1$

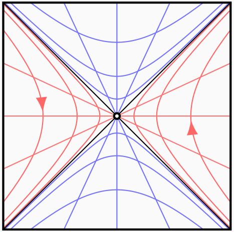
$$\hat{Z}[\beta] \equiv \int_{M\times S^1} \mathcal{D}\phi \, e^{-S[\phi]} = \mathrm{Tr}\, e^{-\beta H}$$

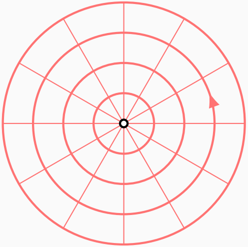
To what extent does this hold for more general manifolds?

For Euclidean $\mathrm{dS}_{d+1} = S^{d+1}$:

$$\log \hat{Z}[\beta] = \log \underbrace{\hat{Z}_{\text{bulk}}}_{\mathrm{Tr}\, e^{-2\pi H_S}} - \log \hat{Z}_{\text{edge}}$$

Euclidean = Thermal?





- H_S has a *fixed point* on S^{d+1}
  - → Bifurcate horizon in Lorentzian dS_{d+1}
- Spatial slicing not smooth at the fixed point
  - → Correction from modes near the horizon

Quantum fields in dS_3

The one-loop partition function for a bosonic field of mass m and spin s:

$$\log \hat{Z} = \int_0^\infty \frac{dt}{2t}\, e^{-t\nu^2} \sum_{n\geq -1} D^{(4)}_{n,s}\, e^{-t(n+1)^2}$$

- $\nu^2 = L^2 m^2 - (s-1)^2$
- $(n+1)^2 =$ shifted Laplacian eigenvalues
- $D^{(4)}_{n,s} = SO(4)$ degeneracies, which decompose as

$$D^{(4)}_{n,s} = D^{(2)}_s D^{(4)}_n - D^{(4)}_{s-1} D^{(2)}_{n+1} = \underbrace{2(n+1)^2}_{\text{bulk}} - \underbrace{2s^2}_{\text{edge}}$$

Harish-Chandra characters

[Anninos, Denef, Law, Sun; (2020)]

$$\log \hat{Z} = \int_0^\infty \frac{du}{2u}\,\frac{1+e^{-u}}{1-e^{-u}}\Big(\chi_{\text{bulk}}(u) - \chi_{\text{edge}}(u)\Big)$$

Both terms are *Harish-Chandra characters* which generalise the characters of compact to non-compact groups:

$$\chi_{\text{bulk}}(u) \equiv \underbrace{D_s^d\,\frac{q^{\frac{d}{2}+i\nu}+q^{\frac{d}{2}-i\nu}}{(1-q)^d}}_{SO(1,d+1)} \qquad \chi_{\text{edge}}(u) \equiv D_{s-1}^{d+2}\,\underbrace{\frac{q^{\frac{d-2}{2}+i\nu}+q^{\frac{d-2}{2}-i\nu}}{(1-q)^{d-2}}}_{SO(1,d-1)}$$

Goal: Edge modes in QFT → string theory

$$\begin{aligned}\log \hat{Z} &= \int_0^\infty \frac{du}{2u}\,\frac{1+e^{-u}}{1-e^{-u}}\,\big(\chi_{\text{bulk}}(u) - \chi_{\text{edge}}(u)\big)\\ &= \log \hat{Z}_{\text{bulk}} - \log \hat{Z}_{\text{edge}}\end{aligned}$$

Idea:

Sum over $\hat{Z}_{\text{edge}}$ for the entire tower of string states. *Hope*: a modular invariant, UV finite worldsheet $\mathcal{Z}_{\text{edge}}$.

Strategy:

Use a dS horizon with very large radius L as an IR regulator for the Rindler horizon.

Goal: Edge modes in QFT → string theory

[Sun; (2020)]

Bosonic string on $\mathbb{R}^{1,2} \times (S^1)^{23} \to \mathrm{dS}_3 \times (S^1)^{23}$ with large dS radius L.

Sum over all fields, alla Polchinski:

$$\mathcal{Z}_{\text{edge}} := \log \hat{\mathcal{Z}}_{\text{edge}} := \sum_i \log \hat{Z}^i_{\text{edge}}$$

Why dS_3?

- Compact: No infinite Vol as in AdS
- Only massive states contribute to edge ($\chi_{\text{edge}} = 0 \ \forall s$ for massless fields) ⇒ only need principal series reps
- Little group $SO(2)$ is abelian: states labelled by helicity

Warm-up: bulk modes

$$\log \hat{Z}_{\text{bulk}} = \int_0^\infty \frac{dt}{2t}\, e^{-t\nu^2/L^2} \sum_{n\geq 0} \underbrace{2(n+1)^2}_{\text{bulk}}\; e^{-t(n+1)^2/L^2}$$

Let $t \sim \tau_2$ and $p = \frac{n+1}{L}$, then take $L \to \infty$:

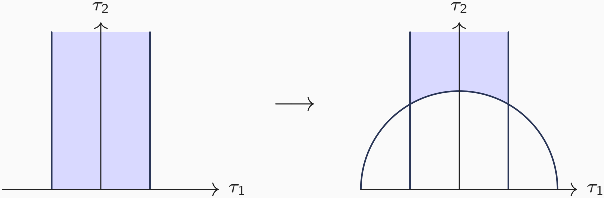
$$\begin{aligned}\log \hat{Z}_{\text{bulk}} &\to L^3 \int_0^\infty \frac{d\tau_2}{\tau_2} \int_0^\infty dp\, p^2 e^{-\pi \ell^2 \tau_2 (p^2+m^2)} \\ &\sim \frac{L^3}{\ell^3} \int_0^\infty \frac{d\tau_2}{\tau_2^{5/2}}\, e^{-\pi \tau_2 \ell^2 m^2}\end{aligned}$$

On-shell condition: $\frac{1}{2}\ell^2 m^2 = N + \tilde{N} - 2 + \frac{1}{2R^2} \sum_{r=1}^{23} n_r^2 \ell^2$

Warm-up: bulk modes

$$\mathcal{Z}_{\text{bulk}} := \sum_i \log \hat{Z}^i_{\text{bulk}} \sim \frac{L^3 R^{23}}{\ell^{26}} \int_{\mathcal{S}} \frac{d^2\tau}{\tau_2^2} \frac{1}{\tau_2^{12}\,|\eta(\tau)|^{48}}$$

Strip $\mathcal{S} \to$ fundamental domain $\mathcal{F}$:



Modular averaging

Need to take

$$\int_{\mathcal{S}} Z(\tau) \to \int_{\mathcal{F}} Z^{\mathsf{inv}}(\tau)$$

where

$$Z^{\mathsf{inv}}(\tau) \;=\; \sum_{\gamma\in\Gamma\backslash SL(2,\mathbb{Z})} Z(\gamma\cdot\tau)$$

For the cosmological constant

$$\Lambda^{\mathsf{inv}}(\tau) \;=\; \sum_{\gamma\in SL(2,\mathbb{Z})\backslash SL(2,\mathbb{Z})} \Lambda(\gamma\cdot\tau) \quad = \Lambda(\tau)$$

Edge modes in string theory

The edge s^2 term, after summing over species:

$$\mathcal{Z}_{\text{edge}} \sim -\frac{L\,R^{23}}{\ell^{24}} \int_0^\infty \frac{d\tau_2}{\tau_2^{13}} \sum_i s_i^2\, e^{-\pi\ell^2\tau_2 m_i^2}$$

Combine 2 of the worldsheet bosons as $X = X_1 + iX_2$:

$$\sum_i s_i^2 \;\longrightarrow\; \mathrm{tr}\left[\tfrac{1}{2}(S+\tilde{S})^2\, q^{N-1}\bar{q}^{\tilde{N}-1}\right]$$

S, $\tilde{S}$: spin operators in the 1-2 plane.

Edge modes in string theory

Insert a real chemical potential μ for S:

$$Z^X(\tau,\mu) = \mathrm{tr}_X\left[q^{N-\frac{1}{12}}\, e^{2\pi\mu S}\right]$$

such that

$$\mathrm{tr}_X\left[(S+\tilde{S})^2\, q^{N-1/12}\bar{q}^{\tilde{N}-1/12}\right] = \lim_{\mu\to 0}\, \frac{1}{4\pi^2}\frac{\partial^2}{\partial\mu^2}\; Z^X(\tau,\mu)\overline{Z^X(\tau,\mu)}$$

Expanding,

$$Z^X(\tau,\mu) \approx \frac{1}{\eta(\tau)^2}\left[1+\frac{\pi^2\mu^2}{6}\big(1-E_2(\tau)\big)+\dots\right]$$

Edge modes in string theory

$$\mathcal{Z}_{\text{edge}} \sim \frac{LR^{23}}{\ell^{24}} \int_{\mathcal{S}} \frac{d^2\tau}{\tau_2^2} \Lambda(\tau) \left(\tau_2 \hat{E}_2(\tau) + \tau_2 \overline{\hat{E}_2(\tau)}\right)$$

where $\hat{E}_2(\tau) = E_2(\tau) - \frac{3}{\pi\tau_2}$ is a modular form of weight 2.

- Integrand still on the strip $\mathcal{S}$, not yet over $\mathcal{F}$
- Need modular averaging:

$$Z^{\text{inv}}(\tau) \; = \sum_{\gamma \in \Gamma \backslash SL(2,\mathbb{Z})} Z(\gamma \cdot \tau)$$

Modular averaging

Map $\mathcal{S} \to \mathcal{F}$: Coset elements $\Gamma \backslash SL(2,\mathbb{Z}) \leftrightarrow$ coprime (c,d):

$$\mathcal{Z}_{\text{edge}} \sim \frac{LR^{23}}{\ell^{24}} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \Lambda(\tau) \Big(F(\tau)\, \hat{E}_2(\tau) + \text{c.c.} \Big)$$

where

$$F(\tau) = \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ (c,d)=1}} \frac{(c\tau+d)^2}{|c\tau+d|^2}\, \tau_2$$

- Formally a modular form of weight -2
- But: the sum does not converge
- Need a regularisation compatible with modular invariance

Modular averaging

Map $\mathcal{S} \to \mathcal{F}$: Coset elements $\Gamma \backslash SL(2,\mathbb{Z}) \leftrightarrow$ coprime (c,d):

$$\mathcal{Z}_{\text{edge}} \sim \frac{LR^{23}}{\ell^{24}} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \Lambda(\tau) \Big(F(\tau)\, \hat{E}_2(\tau) + \text{c.c.} \Big)$$

where

$$F(\tau) = \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ (c,d)=1}} \frac{(c\tau+d)^2}{|c\tau+d|^2}\, \tau_2$$

Introduce a parameter s:

$$F(\tau,s) = \tau_2 \sum_{(c,d)=1} \frac{(c\tau+d)^2}{|c\tau+d|^2} \frac{\tau_2^s}{|c\tau+d|^{2s}}$$

Modular dimensional regularisation

Behaviour of

$$F(\tau, s) = \tau_2 \sum_{(c,d)=1} \frac{(c\tau + d)^2}{|c\tau + d|^2} \frac{\tau_2^s}{|c\tau + d|^{2s}}$$

can be mapped to that of a non-holomorphic Eisenstein series:

$$E_0(\tau, s) = \frac{1}{2} \sum_{(c,d)=1} \frac{\tau_2^s}{|c\tau + d|^{2s}}$$

Kronecker limit formula

The behaviour near $s = 0$ is given by the Kronecker limit formula:

$$E_0(\tau, s) = \frac{\pi}{s-1} - \pi \log\bigl(\tau_2\,|\eta(\tau)|^4\bigr) + \dots$$

Result:

$$F(\tau) = -\frac{\pi^3}{9}\,\tau_2^2\,\overline{\hat{E}_2(\tau)}$$

A finite, weight -2 modular form.

Main result

String edge partition function:

$$\mathcal{Z}_{\text{edge}} \sim \frac{LR^{23}}{\ell^{24}} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \, \Lambda(\tau) \left(\tau_2^2 \, \hat{E}_2(\tau) \, \overline{\hat{E}_2(\tau)} \right)$$

- Manifestly modular invariant
- $\hat{E}_2(\tau)$ has a q-expansion $\Rightarrow$ amenable to a state-counting interpretation

Comments

$$\mathcal{Z}_{\text{edge}} \sim \frac{LR^{23}}{\ell^{24}} \int_{\mathcal{S}} \frac{d^2\tau}{\tau_2^2} \Lambda(\tau) \left(\tau_2 \hat{E}_2(\tau) + \tau_2 \overline{\hat{E}_2(\tau)} - 2\tau_2 \right)$$

We dropped the constant term

- Upon modular averaging, this term is divergent
- dS slightly off-shell at subleading order in L

AdS$_3$ analogue: Similar structure; need to regularise infinite volume; string EOM can be satisfied.

Summary

- Sphere partition function = thermal partition function − edge correction
- Edge contribution = co-dim 2 Harish-Chandra character
- Sum over the bosonic string tower in $\mathrm{dS}_3 \times (S^1)^{23}$
- Modular averaging ⇒ modular invariant, UV finite

Result

$$\mathcal{Z}_{\text{edge}} \sim \frac{LR^{23}}{\ell^{24}} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \, \Lambda(\tau) \, \tau_2^2 \left| \hat{E}_2(\tau) \right|^2$$

Outlook

Result

$$\mathcal{Z}_{\text{edge}} \sim \frac{LR^{23}}{\ell^{24}} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \, \Lambda(\tau) \, \tau_2^2 \left| \hat{E}_2(\tau) \right|^2$$

- AdS$_3$ analogue: on-shell, but needs Vol regularisation
- State-counting interpretation?
- Superstring version?
- Entanglement entropy: $\mathcal{Z}_{\text{bulk}} + \mathcal{Z}_{\text{edge}}$ contributions

Questions?