Wigner negativity, random matrices and Gravity

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Motivation

- The miracle of AdS/CFT is that it allows us to compute some aspects of the strong-coupling, large-N dynamics of the boundary theory.
- For instance, the time evolution of a generic black hole state is hopelessly complicated by the fact that the wavefunction spreads over an exponential number of states.
- But gravity provides an efficient set of variables to describe the strongly-coupled boundary dynamics.
- Given the sucess of AdS/CFT, it is natural to ask whether we can derive it from some underlying fundamental principles.
- Can we construct measures which can be used to probe the emergence of semi-classical bulk spacetime?

Motivation

- In this talk, we will take inspiration from the theory of quantum information and computation.
- The Gottesman-Knill theorem and related work on stabilizer quantum computation identifies quantum circuits that can be simulated efficiently on a classical computer. [Gottesman '98, Aaronson and Gottesman '04, Mari & Eisert '12, Veitch et al '12]
- This construction uses a discrete version of the Wigner function.
- Here, we will argue that this formalism may be well-suited in looking for efficient classical variables for quantum dynamics.

Review of discrete Wigner function

Review: Wigner function

- In standard quantum mechanics, the Wigner function for a state ψ is a *quasi-probability* distribution in phase space:

$$W_\psi(q,p) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dy\,\langle q - \frac{y}{2}|\psi\rangle\langle\psi|q + \frac{y}{2}\rangle\, e^{-ipy}.$$

- Equivalently, one can write:

$$W_\rho(q,p) = \frac{1}{2\pi}\mathrm{Tr}\left(\rho\widehat{A}(q,p)\right),$$

where $\widehat{A}(q,p)$ are Fourier transforms of displacement operators:

$$\widehat{A}(q,p) = \int \frac{dp'\,dq'}{2\pi} e^{i(qp'-pq')} e^{i(p'\widehat{q}-q'\widehat{p})},$$

$$\langle q'|\widehat{A}(q,p)|q''\rangle = \delta(q - \frac{q'+q''}{2})e^{-ip(q'-q'')}.$$

Review: Wigner function

- The Wigner function is real, and it is normalized:

$$\int \frac{dpdq}{2\pi} W_\psi(q,p) = 1,$$

  and integrating it over p (or q) gives the probability density in q (or p).
- However, it is not a probability distribution in general – it can take on negative values.
- States for which the Wigner function is everywhere positive may be regarded as *classical* states (more on this below).
- For instance, a pure state has a positive Wigner function if and only if it is Gaussian (i.e., a generalized coherent state).

  [Hudson '74, Soto & Claverie '83]

Review: Discrete Wigner function

- So far, we have discussed the case of $L^2(\mathbb{R})$, but Wigner functions can also be defined for finite dimensional Hilbert spaces [Wooters '87, Gibbons, Hoffman & Wooters '04, Gross '06 ...].
- For a Hilbert space of dimension D, the phase space is taken to be the lattice $\mathcal{P} = \mathbb{Z}_D \times \mathbb{Z}_D$.
- The formalism works best when D is prime, but can be generalized to arbitrary D.

Review: Discrete Wigner function

- Let $\{|q\rangle\}_{q=0}^{D-1}$ be a choice of an *ordered*, orthonormal basis.
- With respect to this basis, we define exponentiated position and momentum operators as:

$$\widehat{Z}(p)|q\rangle = e^{\frac{2\pi i p q}{D}}|q\rangle, \quad \widehat{X}(q)|q'\rangle = |(q' + q) \bmod D\rangle.$$

- One then defines displacement operators and the corresponding Wigner function in direct analogy with the continuous case:

$$\widehat{A}(q,p) = \sum_{q',q''=0}^{D-1} \tilde{\delta}_{2q,q'+q''} e^{-\frac{2\pi i (q'-q'')p}{D}} |q'\rangle\langle q''|,$$

$$W_\rho(q,p) = \frac{1}{D}\mathrm{Tr}\left(\rho\,\widehat{A}(q,p)\right).$$

Review: Discrete Wigner function

- The discrete Wigner function is also real and normalized:

$$\sum_{q,p=0}^{D-1} W_{\psi}(q,p) = 1,$$

  and summing it over p (or q) gives the probability of q (or p).
- But as before, the Wigner function is not a probability distribution, in that it can take negative values.
- States with positive Wigner functions can be thought of as being classical in the following sense:

Gottesman-Knill theorem

Any quantum circuit which starts with a Wigner positive state and only involves stabilizer (i.e., positivity preserving) operations can be simulated efficiently on a classical computer. [Aaronson and Gottesman '04, Mari & Eisert '12, Veitch et al '12]

Wigner negativity

- For a general state ψ, the negativity of the Wigner function, defined as

$$\mathcal{N}_\psi = \sum_{q,p=0}^{D-1} |W_\psi(q,p)|$$

  can be thought of as a measure of "stabilizer complexity".
- It is a monotone under stabilizer operations [Veitch et al '14].
- Intuitively, one can regard it as a measure of the complexity of simulating the quantum circuit on a classical computer [Stahlke '14, Pashayan et al '15].

Wigner negativity and Uncertainty

- Wigner negativity is also related to quantum uncertainty:

$$S_{1/2}(q) \geq \log \mathcal{N}_\psi,$$
$$S_{1/2} = 2\log \sum_q P_q^{1/2}, \quad P_q = |\langle q|\psi\rangle|^2.$$

  where $S_{1/2}(q)$ is the $1/2$-Renyi entropy of the probability distribution in the q-basis.

- More generally:

$$\min\left(S_{1/2}(q), S_{1/2}(p)\right) \geq \log \mathcal{N}_\psi.$$

- Thus, Wigner negativity necessarily implies some amount of quantum spreading in phase space.

Negativity growth under time evolution

Minimizing negativity growth

- When does the time evolution of a state in a quantum system admit a semi-classical description?
- Inspired by results from quantum information theory, we may say that this happens when the Wigner negativity remains small along time evolution.
- However, recall that the Wigner function is defined with respect to an ordered basis.

So, given an initial state ψ_0 and the time evolution operator e^{-itH}, our task is to find an ordered basis for the Hilbert space such that the Wigner negativity growth of the state under time-evolution is "minimized".

Minimizing negativity growth

Claim

The early time Wigner negativity growth is minimized by the Krylov basis $\mathcal{K}$ [Basu, Ganguly, Nath & OP '24].

Krylov basis

- The Krylov basis is obtained by orthonormalizing the set of states $\psi_0, H\psi_0, H^2\psi_0 \cdots$:

$$|0\rangle_{\mathcal{K}} = |\psi_0\rangle,$$

$$|1\rangle_{\mathcal{K}} = \frac{1}{\sqrt{N_1}} \left(H|\psi_0\rangle - \langle 0_{\mathcal{K}}|H|\psi_0\rangle |0\rangle_{\mathcal{K}} \right),$$

$$|2\rangle_{\mathcal{K}} = \frac{1}{\sqrt{N_2}} \left(H^2|\psi_0\rangle - \langle 0_{\mathcal{K}}|H^2|\psi_0\rangle |0\rangle_{\mathcal{K}} - \langle 1_{\mathcal{K}}|H^2|\psi_0\rangle |1\rangle_{\mathcal{K}} \right),$$

- The Krylov basis is known to minimize the "spread of the wavefunction" [Balasubramanian et al '22].
- The same idea has also appeared previously in the context of operator spreading [Parker et al '18, Swingle et al '20, Rabinovici et al '21].

Minimizing negativity growth: perturbative argument

- By minimizing the early time negativity growth, we mean the following: if we wish to minimize the negativity at $t = 0$, we can simply take ψ_0 to be a basis vector, and without loss of generality, we take it to be the 0th basis vector.

Minimizing negativity growth: perturbative argument

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- Now, for any choice of the first basis vector, the coefficient of the linear in time growth of the negativity is always larger than the coefficient of the linear in time growth in the Krylov basis.

Minimizing negativity growth: perturbative argument

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- Now, for any choice of the first basis vector, the coefficient of the linear in time growth of the negativity is always larger than the coefficient of the linear in time growth in the Krylov basis.
- Similarly, for any basis which agrees with the Krylov basis up to the mth vector, but differs at $m + 1$, the coefficients in the Taylor approximation of the Wigner function agree between the two bases up to t^m, but at $O(t^{m+1})$, the negativity in the Krylov basis is smaller than any other such basis.

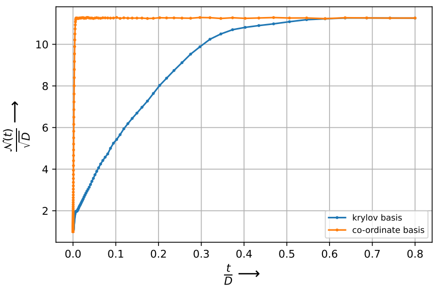
Negativity growth in random matrix theory

- The above argument was perturbative; we would like to go beyond perturbation theory and study the finite time behavior of Wigner negativity.
- In order to make progress, we will have to resort to specific models. To begin with, consider random matrix theory.
- We will choose a Hamiltonian (i.e., a $D \times D$ Hermitian matrix) from the Gaussian unitary ensemble. Chaotic systems are expected to show random matrix theory behavior, so we expect our analysis to apply in such systems.
- The initial state must also be sufficiently generic w.r.t the Hamiltonian.

Negativity growth in random matrix theory

Claim

The Wigner negativity w.r.t a generic basis grows rapidly and saturates to an exponentially large value within an O(1) amount of time evolution. On the other hand, in the Krylov basis, the Wigner negativity grows gradually and takes an exponential amount of time to saturate.



Negativity growth in generic basis

- We can take the basis $\{|i\rangle\}$ to be the one w.r.t which the Hamiltonian is a random matrix.
- Recall that the initial state ψ_0 will be the first basis vector $|0\rangle$ in this basis.
- The averaged Wigner negativity as a function of time

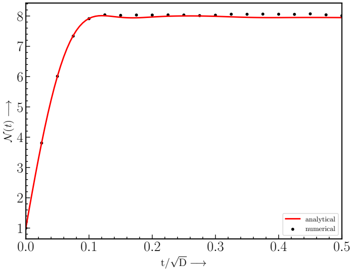
$$\overline{\mathcal{N}(t)} = \frac{1}{Z}\int dH\, e^{-D\,\mathrm{Tr}\,H^2}\mathcal{N}_{\psi_t}, \quad \psi_t = e^{-itH}\psi_0.$$

can be computed using techniques from Haar integration:

$$\overline{\mathcal{N}(t)} = S + \sqrt{\frac{2D}{\pi}}\sqrt{1-S^2} + O(1/\sqrt{D}),$$

$$S(t) = \overline{|\langle\psi_0|e^{-itH}|\psi_0\rangle|^2}$$

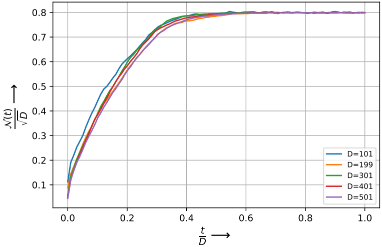
Negativity growth in generic basis



- In particular, $S(t)$ decays away from 1 in an $O(1)$ amount of time evolution.
- So, we see that the negativity grows rapidly and saturates to its maximum value of $\sqrt{\frac{2D}{\pi}}$ in $O(1)$ time.

Negativity growth in the Krylov basis

- On the other hand, the negativity in the Krylov basis grows gradually (power law) for a time of $O(D)$, then saturates to a final value $\sqrt{\frac{2D}{\pi}}$.



Negativity growth in the Krylov basis

- We can understand the slow growth in the Krylov basis analytically.
- The Krylov basis has the special property that it tridiagonalizes the Hamiltonian:

$$H = \begin{pmatrix} a_0 & b_1 & 0 & 0 & \cdots & 0 \\ b_1 & a_2 & b_2 & 0 & \cdots & 0 \\ 0 & b_2 & a_3 & b_3 & \cdots & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & \cdots & b_{n-1} & a_n \end{pmatrix}$$

- The average values of the Lanczos coefficients in GUE are known. In the large D limit, one finds [Balasubramanian et al, '22]

$$\overline{a_n} = 0, \ \ \overline{b_n} = 1, \cdots (D \to \infty, n \text{ fixed})$$

and the variances are $O(1/\sqrt{D})$.

Negativity growth in the Krylov basis

- Crucially, one can argue that the time evolution in this basis stays confined to an $O(1)$ subspace.
- So for any $O(1)$ time, we get a simple *effective* Hamiltonian in the Krylov basis:

$$H_{\text{eff}}|n\rangle = |n-1\rangle + |n+1\rangle.$$

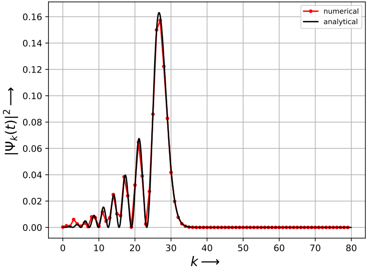
- This Hamiltonian is easily diagonalized:

$$H_{\text{eff}}|\theta\rangle = 2\cos\theta|\theta\rangle, \quad \langle n|\theta\rangle = \sqrt{\frac{2}{\pi}}\sin\left[(n+1)\theta\right].$$

- We can easily compute the time evolution of the initial state:

$$\langle n|e^{-itH_{\text{eff}}}|0\rangle = i^n \frac{(n+1)}{t} J_{n+1}(2t).$$

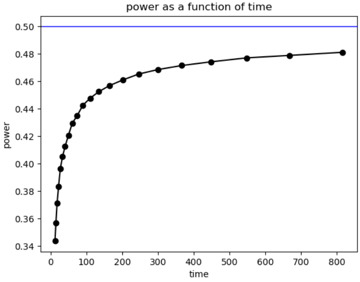
Negativity growth in the Krylov basis



- The wavefunction is *localized* in the region $n \leq 2t$, and decays exponentially beyond. We can use this to bound the growth of Wigner negativity.
- This is enough to argue that the negativity cannot grow faster than $t^{1/2}$.

Negativity growth in the Krylov basis

We actually expect that at sufficiently late time (but still not scaling with D), the negativity grows as $t^{1/2}$.



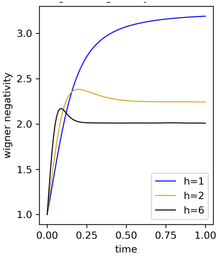
- Analytical evidence for this comes from evaluating the Wigner function in a large-t saddle point approximation.
- In fact, we expect that the $t^{1/2}$ growth is generally true of any single-cut matrix model.

Negativity growth in other models: 2d CFT

- Consider the TFD state in some 2d CFT and excite it by acting on one side with an operator of dimension Δ:

$$\psi_0 = O_\Delta |TFD\rangle_\beta .$$

- Under time evolution by $(H_L - H_R)$, and one finds that the negativity quickly saturates to an $O(1)$ constant.



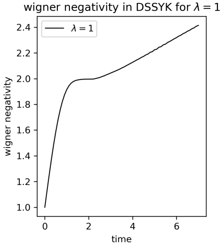
- Furthermore, this constant approaches 2 as $\Delta \to \infty$.

Negativity growth in 2d CFT

- This might be a little suprising as the state is after all spreading with time in the Krylov space.
- But it turns out that the wavefunction becomes peaked in the conjugate momentum basis.
- For $t \gg \beta$, the Wigner function looks like a classical lump moving in q.
- As $\Delta \to \infty$, the lump approaches a Gaussian.
- In this sense, Wigner negativity is a good probe for the emergence of semi-classicality.

Negativity growth in DSSYK

- We also studied the growth of Wigner negativity in the double-scaled SYK model, where the initial state is taken to be the infinite-temperature TFD state.
- In that case, there is a critical time t_* such that for $t < t_*$, the negativity remains contant in time, while for $t \gg t_*$ it grows as $t^{1/2}$.



- Thus, the Wigner function behaves classically for small times, and then develops semi-classical corrections.
- The $t^{1/2}$ behavior at late times matches our expectation from random matrix theory.

Epilogue

- Recall that in the RMT case, we found the effective Hamiltonian in the Krylov basis:

$$H_{\text{eff}}|n\rangle = |n+1\rangle + |n-1\rangle,$$

which was valid for time evolution for $O(1)$ times.

- In this basis, the dynamics is effectively *confined* to a small Hilbert space with n of $O(1)$ in the $D \to \infty$ limit.
- It is tempting to propose this as a toy model for semi-classical bulk emergence.
- Indeed, this particular Hamiltonian is simply the $q \to 0$ limit of q-deformed JT gravity, but the same phenomenon happens at all q. [Berkooz et al, Lin, Rabinovici et al '23, Blommaert et al '24].

Epilogue

- From this perpective, gravity is a *low-complexity* effective theory., where the relevant notion of complexity seems to be the complexity of classical simulation.
- Note that the effective Hamiltonian has corrections at finite D coming from the fact that the Lanczos coefficients $\{a_n, b_n\}$ were actually random variables with variances suppressed by $O(\frac{1}{\sqrt{D}})$.
- These corrections can be accounted for by EFT operators but with random coefficients, and should give rise to wormholes.

  [WIP]