Dilaton gravity in box: canonical quantization of JT and sinh models

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WIP with L. Russo (Florence), A. Tarana (Parma), L. Griguolo (Parma), D. Seminara (Florence)

Introduction and motivation

Finite cutoff quantum gravity: define **bulk observables** associated with boundary observers located at finite radial location

Combine with low dimensional (2d) gravity framework

• **Most renowned example: JT gravity**

$\rightarrow$ duality on the disk with boundary reparametrization mode captured by **Schwarzian theory** $\rightarrow$ low-energy limit of SYK model
$\rightarrow$ what happens if we push the boundary clock into the bulk?

• 2 other solvable theories: **Liouville and complex Liouville gravity**

$\rightarrow$ non-critical string theories given by **double 2d Liouville CFT**
$\rightarrow$ Dilaton gravity perspective: through field redefinition

$$\phi = b^{-1}\rho - b\pi\Phi\,, \quad \chi = b^{-1}\rho + b\pi\Phi\,,$$

mapped to 2d dilaton gravities with **sinh/sine potential**
$\rightarrow$ **asymptotic dualities** with **DSSYK/q-Schwarzian deformed?**

Plan of the talk

JT gravity

- Classical JT gravity as a scattering problem at finite cutoff
- Canonical quantization and partition function
- Embedding into $\mathrm{SL}(2, \mathbb{R})$ group theory
- Exact and semiclassical two-point function

Sinh dilaton gravity (results only)

- Effective geometry Hamiltonian formulation
- Canonical quantization and partition function
- Quantum group $\mathrm{SL}_q(2, \mathbb{R})$ perspective

Outlook

JT gravity on shell

$$S_{\text{JT}}[g,\Phi] = \frac{1}{2}\left[\int_M \mathrm{d}^2x\sqrt{-g}\,\Phi\,(R+2) + 2\int_{\partial M}\mathrm{d}u\sqrt{|h|}\,\Phi\,(K-1)\right]$$

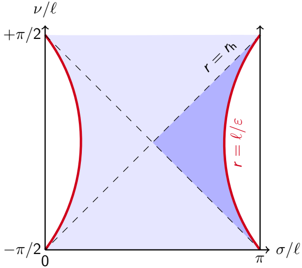
Classical solutions in the Schwarzschild gauge

$$\mathrm{d}s^2 = (r_h^2 - r^2)\mathrm{d}t^2 - \frac{1}{r_h^2 - r^2}\mathrm{d}r^2, \qquad \Phi = r$$

Standard formulation: $\epsilon \to 0$

$$\Phi\Big|_{\partial M} = \Phi_b \sim 1/\epsilon, \qquad g_{\mu\nu}\,dx^\mu dx^\nu\Big|_{\partial M} = \text{const.}$$

→ **Goal**: keep Φ_b finite
→ Consider the global extension of AdS_2
→ **Two-dimensional phase space**



What are the natural **phase space variables** in the two-sided finite cutoff description?

Diff invariant quantity $\rightarrow$ **proper time** $\tau^{\mathrm{L/R}}$ of the boundaries

$$\tau^{\mathrm{L/R}} \equiv \sqrt{\Phi_b^2 - \Phi_h^2}\, t^{\mathrm{L/R}},$$

Killing vector fields $\zeta_{\mathrm{L/R}} \equiv \partial_{\tau^{\mathrm{L/R}}} = \frac{1}{\sqrt{\Phi_b^2 - \Phi_h^2}} \partial_{t^{\mathrm{L/R}}}$

The corresponding **Hamiltonians** $\rightarrow$ **conserved charges** associated with translations in $\tau^{\mathrm{L/R}}$.

$$H^{\mathrm{L/R}} = \zeta^{\alpha}_{\mathrm{L/R}} \zeta^{\beta}_{\mathrm{L/R}} T^{\mathrm{L/R}}_{\alpha\beta} = \boxed{\Phi_b - \sqrt{\Phi_b^2 - \Phi_h^2}},$$

where $T^{\mathrm{L/R}}_{\alpha\beta}$ is the Brown–York stress tensor on each boundary:

$$T^{\mathrm{L/R}}_{\alpha\beta} \equiv -\frac{2}{\sqrt{|h|}} \frac{\delta S_{\mathrm{JT}}}{\delta h^{\alpha\beta}} = \left(n^{\mu} \nabla_{\mu} \Phi - \Phi\right)\Big| h_{\alpha\beta}$$

$H = \frac{H^L + H^R}{2}$ is **quasi-local Hamiltonian** conjugate to **Tolman temperature** $\beta_{\mathrm{T}} = \beta$.

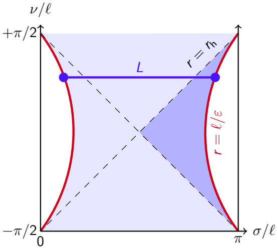
Define a finite cutoff **symplectic measure** $\omega = \mathrm{d}\tau \wedge \mathrm{d}H$ on phase space [1]

[1] Harlow, Jafferis.

JT gravity as a scattering problem

Natural variable: **geodesic length** of a **two-sided Cauchy slice** at finite cutoff:

$$L = 2\sinh^{-1}\left[\frac{\sqrt{\Phi_b^2 - \Phi_h^2}}{\Phi_h}\cosh(\Phi_h t)\right]$$



Canonical **change of variables**: $\omega = \mathrm{d}L \wedge \mathrm{d}P$

$$H(L,P) = \Phi_b - \sqrt{\Phi_b^2 - \left(P^2 + \frac{\Phi_b^2}{\cosh^2(L/2)}\right)}$$

We focus on the **Pöschl–Teller Hamiltonian** inside the square root

$$\boxed{H_0 = P^2 + \frac{\Phi_b^2}{\cosh^2(L/2)}}$$

As $\Phi_b \gg 1$, $L_{\mathrm{ren}} = L - 2\log(\Phi_b)$, it reduces to standard Liouville Hamiltonian $H_0 \to \frac{1}{2\Phi_b}\left(P^2 + e^{L_{\mathrm{ren}}}\right)$.

Canonical quantization

We propose a **quantum ordering** yielding the **Schrödinger equation**: [2]

$$-\frac{\mathrm{d}^2\psi_s(L)}{\mathrm{d}L^2}+\left(\frac{\nu^2}{\cosh^2(L/2)}-\frac{1}{16\sinh^2(L/2)\cosh^2(L/2)}\right)\psi_s(L)=s^2\psi_s(L).$$

Effective potential with quantum effect $V_{\text{eff}}(L)\overset{L\to 0}{\sim}-\frac{\hbar^2}{4L^2}$.

$$V_{\text{eff}}(L)=\frac{\Phi_b^2}{\cosh^2(L/2)}-\frac{\hbar^2}{16\sinh^2(L/2)\cosh^2(L/2)},$$

which is precisely induced by $b\to 0$ limit of sinh dilaton gravity!

The wavefunction is

$$\boxed{\psi_s(L)=\tanh^{\frac{1}{2}}\left(\frac{L}{2}\right)\cosh^{-2is}\left(\frac{L}{2}\right){}_2F_1\left(\frac{1}{2}+is+i\nu,\frac{1}{2}+is-i\nu,1;\tanh^2\left(\frac{L}{2}\right)\right)}$$

vanishes at $L=0\to$ consistent with the interpretation of L as geodesic length $L\geq 0$.

[2] $\nu\equiv\Phi_b/\hbar$, $s\equiv\Phi_h/\hbar$

As $\nu \gg 1$, $L_{\rm ren} = L - 2\log(\nu)$ it reduces to $\psi_s(L) \to K_{2is}\left(2e^{-L_{\rm ren}/2}\right)$

Near $L = 0$, actually

$$\psi_s(L) \overset{L\to 0}{\sim} \sqrt{L} + \alpha\sqrt{L}\log L.$$

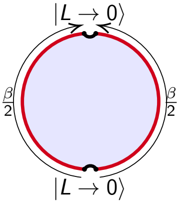
we select $\alpha = 0$ because of embedding into **group theory** .

Orthogonality condition:

$$\int_0^\infty \mathrm{d}L\,\psi_s(L)\psi_{s'}(L) = \frac{\delta(s-s')}{\rho(s)} \qquad \boxed{\rho(s) = \frac{s\sinh(2\pi s)}{\cosh(2\pi s) + \cosh(2\pi\nu)}}$$

Partition function $\to$ **Euclidean transition amplitude** between states of zero geodesic length $|L = 0\rangle$.

$$Z(\beta) = \langle L = 0|\, e^{-\beta \hat{H}}\, |L = 0\rangle =$$



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The state $|L = 0\rangle$ replaces $|L_{\rm ren} = -\infty\rangle$!

Partition function

Since for small-L $\psi_s(L) \overset{L\to 0}{\sim} \sqrt{L}$, we extract the leading vanishing behaviour:

$$Z(\beta) = \left(\lim_{L\to 0} L\right) \int_0^\infty \mathrm{d}s \, \frac{s \sinh(2\pi s)}{\cosh(2\pi s) + \cosh(2\pi \nu)} e^{-\beta E(s)}.$$

where $E(s) = \nu - \sqrt{\nu^2 - s^2}$.

Issue: what happens when $s > \nu$? Black hole horizon Φ_h becomes larger than cutoff Φ_b. **We will come back to this later.**

Equivalent form

$$Z(\beta) = \left(\lim_{L\to 0} L\right) \int_0^\infty \mathrm{d}s \, s \sinh(2\pi s) \, \Gamma\left(\frac{1}{2} \pm is \pm i\nu\right) e^{-\beta E(s)}.$$

Infinite cutoff two-point function with a fixed energy label beyond the boundary particle trajectory and $\Delta = 1/2$.

Comparison with previous work

Our result encodes **two sources of dependence on** ν (Φ_b):

- Deformation of the **Boltzman factor** $\beta E(s) = \beta\left(\nu - \sqrt{\nu^2 - s^2}\right)$

$\rightarrow$ at large ν reduces to $\beta_{\mathrm{JT}} s^2$ with $\beta_{\mathrm{JT}} = \beta/\nu$

$\rightarrow$ corresponds to the **perturbative branch of $T\bar{T}$ spectrum** [3]

$\rightarrow$ in our case $T\bar{T}$ does not imply $\hat{H}_\lambda = \nu - \sqrt{\nu^2 - \left(\hat{P}^2 + e^{-\hat{L}_{\mathrm{ren}}}\right)}$

- Deformation of the **spectral density/measure**

$$\rho(s) = \frac{s \sinh(2\pi s)}{\cosh(2\pi s) + \cosh(2\pi\nu)} \rightarrow s \sinh(2\pi s) + \mathcal{O}\left(e^{-2\pi\nu}\right)$$

$\rightarrow$ **nonperturbative corrections** in the cutoff!

$\rightarrow$ consistent with identification parameter μ of **universal cover of** $\mathrm{SL}(2,\mathbb{R})$ as $\mu = iB$ [4] and $B = 1/\epsilon$ in **JT=particle in a magnetic field** [5]

[3] Gross, Kruthoff, Rolph, Shaghoulian. Iliesiu, Kruthoff, Turiaci, Verlinde
[4] Iliesiu, Pufu, Verlinde, Wang
[5] Yang, Kitaev

A group theory interpretation

In the infinite cutoff case, using the **Gauss-Euler decomposition** of $\mathrm{SL}(2,\mathbb{R})$ group element $g = e^{\gamma F} e^{\phi H} e^{\beta E}$ [6]

$\rightarrow$ fix eigenvalues of **mixed parabolic generators** E, F

$\rightarrow$ reduced Casimir equation $\langle \eta | \, g \, \mathcal{C} \, | \mu \rangle$ becomes **Liouville quantum mechanics** $-\frac{d^2}{d\phi^2} - \mu\eta e^{-2\phi}$

If instead we use **KAK decomposition** $g = e^{\theta_1 (E-F)} e^{LH} e^{\theta_2 (E-F)}$

$\rightarrow$ fix eigenvalues of **elliptic generator** $E - F$

$\rightarrow$ reduced Casimir operator $\langle m | \, g \, \mathcal{C} \, | n \rangle$ becomes

$$-\frac{d^2}{dL^2} + V_{m,n}(L) \qquad V_{m,n}(\phi) = \frac{(m-n)^2 - 1}{4 \sinh^2 L \cosh^2 L} - \frac{mn}{\cosh^2 L}$$

Matches with **quantum Pöschl–Teller Hamiltonian** for $m = n = i\nu = i\Phi_b / \hbar$!

[6] Mertens, Blommaert, Verschelde

• Renormalizing the length $L_{\rm ren} = L - 2\log(\Phi_b)$ and using commutation relations

$$e^{\theta_1(E-F)} e^{\frac{1}{2}\log(\phi_b)H} e^{\phi H} e^{\frac{1}{2}\log(\phi_b)H} e^{\theta_2(E-F)}$$

The KAK decomposition reduces to Gauss-Euler as $\Phi_b \gg 1$.

• **Representation matrix element** (with $\lambda = \frac{1}{2} + is$ principal-series)

$$R_{m,m}(g) \propto e^{im(\theta_1+\theta_2)}(1-u)^{\lambda}\, {}_2F_1(\lambda+m, \lambda-m; 1; u), \qquad u = \tanh^2 L/2.$$

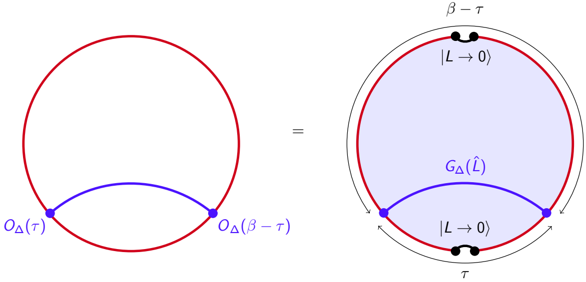
corresponds to the **bulk wavefunction** written above!

• Analytically continuing representation label, **discrete series** are

$$R^{(\Delta)}_{m,n}(u) = u^{\frac{m-n}{2}}(1-u)^{\Delta} P_N^{(m-n,\, 2\Delta-1)}(1-2u)$$

The **lowest-weight coefficient** $\to R^{\Delta}_{0,0} \propto \cosh{(L/2)}^{-2\Delta}$
Analogue of $R^{\Delta}_{0,0} = e^{-\Delta\phi} \to$ This should encode **matter insertion!**

Two-point function



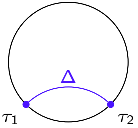
For finite cutoff generalization of $e^{-\Delta \hat{L}_{\text{ren}}}$, we **propose insertion of**

$$G_\Delta(\hat{L}) = \left[\cosh\left(\frac{\hat{L}}{2}\right)\right]^{-2\Delta} \overset{\Phi_b\to\infty}{\sim} 2^{2\Delta} e^{-\Delta \hat{L}} = \Phi_b^{-2\Delta} e^{-\Delta \hat{L}_{\text{ren}}}.$$

same large-cutoff behavior as the standard exponential proposal. The **matrix element** can be computed exactly!

$$\mathcal{I}_{\cosh}(\Delta; s_1, s_2) := \int_0^\infty \psi_{s_1}(L)\, \psi_{s_2}(L)^* \left(\cosh(L/2)\right)^{-2\Delta} \mathrm{d}L.$$

$$\langle O_\Delta(\tau_1, \tau_2)\rangle_\beta =$$



$$= \int_0^\infty \mathrm{d}\mu(s_1)\,\mathrm{d}\mu(s_2) e^{-(\beta-\tau_2+\tau_1)E(s_2)} e^{-(\tau_2-\tau_1)E(s_1)} \times$$
$$\times\, \Gamma(\Delta \pm i s_1 \pm i s_2)\, \mathbb{W}(s_1, \nu; \tfrac{1}{2} + i s_2, \tfrac{1}{2} - i s_2, \Delta - i\nu, \Delta + i\nu)$$

with

$$\mathrm{d}\mu(s) \equiv \frac{2s \sinh(2\pi s)}{\cosh(2\pi s) + \cosh(2\pi\nu)} \mathrm{d}s$$

and $\mathbb{W}(\lambda, x; a, b, c, d)$ is the Wilson function.
$\rightarrow$ **encodes the 6j symbol in infinite cutoff JT**

We checked in the **semiclassical** $G_N \rightarrow 0$ **limit**, it localizes to

$$L_{\mathrm{E}}(\tau, \beta) = 2 \sinh^{-1}\left(\frac{\beta}{2\pi}\left|\sin\left(\pi\frac{\tau}{\beta}\right)\right|\right),$$

First finite-cutoff 2pt function proposal to do so!

What happens when $\Phi_h > \Phi_b$?

- $T\bar{T}$ point of view:

$\rightarrow$ often associated to resurgence of **nonperturbative branch** [7]

$\rightarrow$ not fully clear gravitational interpretation

- **by-hand truncation** of the spectrum at $\Phi_h = \Phi_b$

$$\hat{\Pi} e^{-\beta \hat{H}} \hat{\Pi}, \qquad \hat{\Pi} \equiv \int_0^{\Phi_b} s \, |s\rangle \langle s| \,.$$

$\rightarrow$ state $\hat{\Pi}\,|L = 0\rangle$ is **not a sharply localized** zero-length state anymore

- closer look at the Penrose diagram in global coordinates.

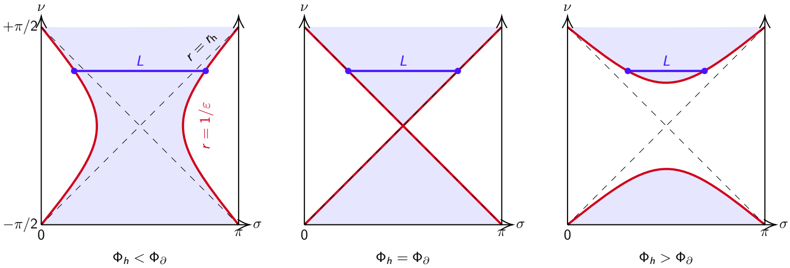
$$\mathrm{d}s^2 = \frac{-\mathrm{d}\nu^2 + \mathrm{d}\sigma^2}{\sin^2(\sigma)}, \qquad \Phi = \Phi_h \frac{\cos(\nu)}{\sin(\sigma)},$$

Depending on Φ_b, Φ_h $\rightarrow$ **timelike, null or spacelike boundary curves.**
We can still compute the **conserved charge** associated to $\zeta_{\mathrm{L,R}}$

$$E(s) = \begin{cases} \Phi_b - \sqrt{\Phi_b^2 - s^2}, & s < \Phi_b, \\ \Phi_b + \sqrt{s^2 - \Phi_b^2}, & s > \Phi_b. \end{cases}$$

[7] Iliesiu, Pufu, Verlinde, Wang. Griguolo, Panerai, Papalini, Seminara

Our proposal: going beyond the horizon



Partition function with **piecewise defined energy** function:

$$Z(\beta) = \int_0^\infty \mathrm{d}s \, \frac{s \sinh(2\pi s)}{\cosh(2\pi s) + \cosh(2\pi \nu)} e^{-\beta E(s)}.$$

$$E(s) = \begin{cases} \nu^2 - \sqrt{\nu^2 - s^2}, & s < \nu, \\ \nu^2 + \sqrt{s^2 - \nu^2}, & s > \nu. \end{cases}$$

How can we make sense of **space-like evolution** behind the horizon?

$\rightarrow$ Possible interpretation in terms of $T\bar{T} + \Lambda$ deformation. [8]

[8] Ahmad, Almheiri, Lin

Sinh dilaton gravity: summary of main results

- Canonical quantization is performed using **finite cutoff length of ERB in the Weyl rescaled geometry** $\mathrm{d}s^2_{\mathrm{eff}} = e^{2\pi b^2 r}\,\mathrm{d}s^2 \rightarrow$ **effective AdS$_2$**[9]

- Fine cutoff Hamiltonian has the same $T\bar{T}$-structure, with

$$\boxed{H_0 = \sqrt{\left(1 + e^{4\pi b^2 \Phi_b} e^{-\Lambda}\right)\left(1 + e^{-\Lambda}\right)}\cosh(\Pi) - e^{2\pi b^2 \Phi_b} e^{-\Lambda}}$$

- Wavefunction is q-hypergeometric function: [10]

$$\boxed{\chi_s(\Lambda) = e^{\Lambda(\frac{1}{2}+i\nu)} F_b\left(\frac{b}{2} + ib\nu + ibs, \frac{b}{2} + ib\nu - ibs, b, -e^{\Lambda - 4\pi b^2 \nu}\right)}$$

- Spectral density from orthogonality:

$$\boxed{\rho(s) = \frac{\sinh\left(2\pi b^2 s\right)\,\sinh(2\pi s)}{\cosh(2\pi s) + \cosh(2\pi\nu)}}$$

- **Reduced $\mathfrak{sl}_q(2,\mathbb{R})$ quantum Casimir** $\langle s_1 |\, g\, \mathcal{C}\, | s_1 \rangle$ with fixed hyperbolic labels $s_1 = \nu$ matches with **finite cutoff Hamiltonian**

[9] Blommaert, Mertens, Papalini.

[10] Ponsot, Teschner, Ip

Outlook

- **First order formulation** of JT gravity at finite cutoff
- Explain the **boundary conditon** $E - F$ **fixed to be** $\propto \Phi_b$ from the bulk
- Physics beyond the horizon in terms of $T\bar{T} + \Lambda$ **deformation**
- Construct finite cutoff amplitudes in terms of **infinite cutoff Feynman rules**
- Explain the **insertion of** $\cosh(L/2)^{-2\Delta}$ from matter coupled to gravity in the bulk
- **OTOC and Lyapunov exponent** at finite cutoff
- **Sine dilaton at finite cutoff**: complexified contour and deformation of DSSYK

Thanks.

Sinh dilaton gravity on shell

$$S = -\frac{1}{2}\int_{\mathcal{M}} d^2x\sqrt{g}\left(\Phi R + \frac{\sinh\left(2\pi b^2\Phi\right)}{\pi b^2}\right) - \oint_{\partial\mathcal{M}} d\tau\,\sqrt{h}\,(\Phi K - \mathrm{c.t.})$$

In the gauge $\Phi = r$, classical solution is:

$$\mathrm{d}s^2 = -\left[\cosh 2\pi b^2 r - \cosh 2\pi b^2\Phi_h\right]\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{\left[\cosh 2\pi b^2 r - \cosh 2\pi b^2\Phi_h\right]}$$

Black hole horizon at $r = \Phi_h$ and **holographic screen** at $r = \Phi_b \to \infty$, where holographic singularity is found $\to$ not standard holographic setup

Quasi-local energy can be computed again

$$H(\Phi_h) = \sqrt{\cosh(2\pi b^2\Phi_b)} - \sqrt{\cosh(2\pi b^2\Phi_b) - \cosh(2\pi b^2\Phi_h)}.$$

Infinite cutoff lesson: canonical quantization can be performed in an effective Weyl rescaled AdS_2, corresponding to the saddle of the Liouville field φ [11]

Natural: finite cutoff quantization in the effective geometry!

[11] Blommaert, Mertens, Papalini.

Weyl rescaled geometry

Effective geometry with **same entropy** and **Hawking temperature**:

$$\mathrm{d}s^2_{\mathrm{eff}} = e^{2\pi b^2 r}\,\mathrm{d}s^2 \qquad \rho(r) = e^{2\pi b^2 r} - \cosh\left(2\pi b^2 \Phi_h\right)$$

Takes the original metric to an **effective AdS_2 black hole**:

$$\mathrm{d}s^2_{\mathrm{eff}} = -(\rho^2 - \rho_h^2(\Phi_h))\,\mathrm{d}t^2 + \frac{\mathrm{d}\rho^2}{\rho^2 - \rho_h^2(\Phi_h)} \qquad \rho_h(\Phi_h) = \sinh\left(2\pi b^2 \Phi_h\right).$$

Fix the boundary location $r = \Phi_b$ in the original geometry

$\rightarrow$ sets a boundary location at $\rho = \rho_b$ **in the effective geometry**:

$$\rho_b = \rho_b(\Phi_b, \Phi_h) = e^{2\pi b^2 \Phi_b} - \cosh\left(2\pi b^2 \Phi_h\right).$$

Using the maps $\rho_b(\Phi_b, \Phi_h)$ and $\rho_h(\Phi_h)$, compute **finite cutoff ERB length** in the effective AdS_2:

$$\sinh\left(\frac{L}{2}\right) = \sqrt{\left(\frac{e^{2\pi b^2 \Phi_b} - \cosh(2\pi b^2 \Phi_h)}{\sinh(2\pi b^2 \Phi_h)}\right)^2 - 1}\,\cosh\left(t \sinh\left(2\pi b^2 \Phi_h\right)\right)$$

Again using $\omega = \mathrm{d}\tau \wedge \mathrm{d}H = \mathrm{d}L \wedge \mathrm{d}P$, one finds for $H_0 = \cosh(2\pi b^2 s)$:

$$H_0 = \sqrt{1 + \frac{e^{4\pi b^2 \Phi_b}}{\sinh^2(L/2)}} \frac{1}{\tanh(L/2)} \cosh\left(2\pi b^2 P \tanh(L/2)\right) - \frac{e^{2\pi b^2 \Phi_b}}{\sinh^2(L/2)}.$$

Double deformation (b, Φ_b) of Liouville Hamiltonian of JT gravity!

As $b \ll 1 \rightarrow$ **Pöschl–Teller Hamiltonian**

$$\cosh(2\pi b^2 \Phi_h) = 1 + \frac{1}{2}\left(2\pi b^2\right)^2 \left(P^2 + \frac{\Phi_b^2}{\cosh^2(L/2)}\right) + \cdots$$

As $\Phi_b \gg 1$, $L_{\mathrm{ren}} = L - 4\pi b^2 \Phi_b \rightarrow$ **Liouville gravity Hamiltonian**

$$\lim_{\Phi_b \to \infty} \cosh(2\pi b^2 \Phi_h) = \sqrt{1 + e^{-L_{\mathrm{ren}}}} \cosh(2\pi b^2 P).$$

Perform a canonical transformation:

$$\Lambda := 2 \ln\left(\sinh \frac{L}{2}\right), \qquad \Pi := P \tanh \frac{L}{2},$$

More pleasant form:

$$\boxed{H_0 = \sqrt{\left(1 + e^{4\pi b^2 \Phi_b} e^{-\Lambda}\right)\left(1 + e^{-\Lambda}\right)} \cosh(\Pi) - e^{2\pi b^2 \Phi_b} e^{-\Lambda}}$$

Schrödinger equation and wavefunction

$$\frac{1}{2}\left(1+e^{-\Lambda+i\pi b^2}\right)\chi_s(\Lambda-2\pi i b^2)+\frac{1}{2}\left(1+e^{4\pi b^2\nu}e^{-\Lambda-i\pi b^2}\right)\chi_s(\Lambda+2\pi i b^2)$$
$$=\left(\cosh\left(2\pi b^2 s\right)+e^{2\pi b^2\nu}e^{-\Lambda}\right)\chi_s(\Lambda).$$

As $b\to 0$, it yields the quantum Pöschl–Teller potential!

In **Fourier** space:

$$\frac{\widetilde{\chi}_s(k+i)}{\widetilde{\chi}_s(k)}=e^{-2\pi b^2\nu}\frac{\sinh\left(\pi b^2(s+k)\right)\sinh\left(\pi b^2(s-k)\right)}{\sinh\left(\pi b^2(k+\nu+i/2)\right)^2}.$$

A solution can be found

$$\chi_s(\Lambda)=\frac{S_b(b)}{S_b(\frac{b}{2}+ib\nu\pm ibs)}\int \mathrm{d}k\, e^{-i\Lambda k}\,\frac{S_b\left(-ibk\pm ibs\right)S_b\left(\frac{b}{2}+ib\nu+ibk\right)}{S_b\left(\frac{b}{2}-ib\nu-ibk\right)}e^{2\pi i b^2\nu k}$$

→ Mellin-Barnes representation of ***q*-hypergeometric function**: [12]

$$\boxed{\chi_s(\Lambda)=e^{\Lambda(\frac{1}{2}+i\nu)}F_b\left(\frac{b}{2}+ib\nu+ibs,\frac{b}{2}+ib\nu-ibs,b,-e^{\Lambda-4\pi b^2\nu}\right)}$$

[12] Ponsot, Teschner, Ip

All the **correct limits** are verified:

As $\nu \to \infty$, with $\Lambda_{\rm ren} = \Lambda - 4\pi b^2 \nu$ it reduces to q-deformed BesselK, i.e. **Liouville gravity wavefunction** [13]

$$\to e^{\pi\nu} \int_{-\infty}^{+\infty} \mathrm{d}k \, e^{-i\Lambda_{\rm ren} k} S_b \left(-ibk \pm ibs\right) e^{i\pi b^2 (k^2 - s^2)}$$

As $b \to 0$, using $S_b(bx) \to \Gamma(x)$ it reduces to **standard JT hypergeometric**

$$\to \tanh\left(L/2\right) \left(\cosh\left(L/2\right)\right)^{1+2i\nu} {}_2F_1\left(\frac{1}{2} + i\nu + is, \frac{1}{2} + i\nu - is, 1; -\sinh^2\left(L/2\right)\right)$$

WKB scattering region: encoded by poles at $k = \pm s$

$$\chi_s(\Lambda) = A_s e^{-i\Lambda s}\left(1 + \mathcal{O}(e^{-\Lambda})\right) + A_{-s} e^{i\Lambda s}\left(1 + \mathcal{O}(e^{-\Lambda})\right) \, .$$

Asymptotic structure of $\chi_s(\Lambda)$ as $\Lambda \to \infty$, with

$$A_s = \frac{S_b\left(-2ibs\right)}{S_b\left(\frac{b}{2} - ib\nu - ibs\right) S_b\left(\frac{b}{2} + ib\nu - ibs\right)} \, e^{2\pi i b^2 \nu s} \, .$$

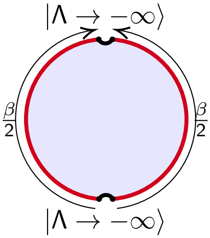
[13] Mertens, Turiaci, Fan

Spectral density and partition function

We can infer the orthogonality condition $\int \mathrm{d}\Lambda\, \chi_s^*(\Lambda)\chi_{s'}(\Lambda) = \frac{\delta(s-s')}{\rho(s)}$

$$\boxed{\rho(s) = |A_s|^2 = \frac{\sinh\left(2\pi b^2 s\right)\, \sinh(2\pi s)}{\cosh(2\pi s) + \cosh(2\pi \nu)}}$$

$Z(\beta) =$



$$= \left(\lim_{\Lambda \to -\infty} e^{\Lambda}\right) \int_0^{+\infty} \mathrm{d}s |\psi_s(\Lambda)|^2 e^{-\beta E(s)},$$

$$Z(\beta) = \int_0^{\infty} \mathrm{d}s\, \frac{\sinh\left(2\pi b^2 s\right)\, \sinh(2\pi s)}{\cosh(2\pi s) + \cosh(2\pi\nu)} e^{-\beta E(s)}.$$

$$E(s) = \begin{cases} \cosh\left(2\pi b^2 \nu\right) - \sqrt{\cosh(2\pi b^2 \nu) - \cosh(2\pi b^2 s)}, & s < \nu, \\ \cosh\left(2\pi b^2 \nu\right) + \sqrt{\cosh(2\pi b^2 s) - \cosh(2\pi b^2 \nu)}, & s > \nu. \end{cases}$$

A quantum group perspective

Standard Liouville gravity → **mixed-parabolic Casimir** equation $\mathcal{C}$ (where $T_a^{\phi} f(\phi) = f(\phi + a)$)

$$\frac{1}{2}\left(T^{\phi}_{i\pi b^2} + T^{\phi}_{-i\pi b^2} + \mu\eta e^{-2\phi}\right)\psi_{\mu\eta}(g) = \cosh(2\pi bs)\,\psi_{\mu\eta}(g)$$

and its $b \to 1/b$ dual $\tilde{\mathcal{C}}$. Unique solution is **Whittaker function**:

$$\psi_{\mu\eta}(g) = \int_{\mathcal{C}} d\xi\, \mu^{\frac{1}{b}(i\xi - is)}\eta^{\frac{1}{b}(i\xi + is)} S_b\left(-i\xi \pm is\right) e^{-2i\phi\xi/b}$$

From **mixed-parabolic to hyperbolic** matrix element

$$\langle\eta|\, g\, |\mu\rangle = \int_{-\infty}^{\infty} \mathrm{d}s_1 \mathrm{d}s_2 \langle\eta\, |s_1\rangle \langle s_1|\, g\, |s_2\rangle \langle s_2\, |\mu\rangle$$

Exploiting the Gauss-Euler decomposition $g = g_b(\gamma F) e^{2\phi H} g_b^*(\beta E)$: [14]

$$R_{s_1 s_2}(g) = \frac{1}{\mathcal{N}} \int_0^{+\infty} \mathrm{d}\mu\ \mu^{is_2 + is/b - \frac{1}{2}} g_b^*(\beta\mu) \int_{\mathcal{H}} \mathrm{d}\eta\ \eta^{is_1 - is/b - \frac{1}{2}} g_b(-\gamma\eta)\ \ \psi_{\mu\eta}(\phi)$$

The hyperbolic matrix element $R_{s_1 s_2}(g) \equiv \langle s_1|\, g\, |s_2\rangle$ and the Whittaker function related by **double integral transform**.

[14] Mertens

Acting with **double transform** $\mathcal{L}_\mu \mathcal{L}_\eta$ on the Casimir equation $\langle \eta | \, g \; \mathcal{C} \, | \mu \rangle$

$\rightarrow$ **Reduced Casimir** $\langle s_1 | \, g \; \mathcal{C} \, | s_1 \rangle$ on hyperbolic matrix elements

$$\frac{1}{2}\Big(T^{\phi}_{i\pi b^2} + T^{\phi}_{-i\pi b^2} + (\gamma e^{2\phi}\beta)^{-1}\big(e^{i\pi b^2} e^{-2\pi b s_1} T^{\phi}_{-i\pi b^2} + e^{-i\pi b^2} e^{2\pi b s_1} T^{\phi}_{i\pi b^2} - 2\big)\Big) R_{s_1 s_1}(g)$$
$$= \cosh(2\pi b s) \, R_{s_1 s_1}(g)$$

If we identify hyperbolic element as

$$\gamma e^{2\phi}\beta \equiv e^{\Lambda - 2\pi b^2 \nu}$$

$\rightarrow$**Finite cutoff Hamiltonian derived from the bulk!**

Finally, performing the double integral transform

$$R_{s_1 s_1}(g) = \frac{\gamma^{-is_1 - 1/2}\beta^{-is_1 - 1/2}}{S_b\left(\frac{b}{2} + ibs_1 \pm is\right)} \int d\xi \; \left(\gamma e^{2\phi}\beta\right)^{-i\xi/b} \; S_b\left(-i\xi \pm is\right) \frac{S_b^2\left(\frac{b}{2} + ibs_1 + i\xi\right)}{S_b^2\left(\frac{b}{2} - ibs_1 - i\xi\right)}$$

Finite cutoff wavefunction $\chi_s(\Lambda)$ $\rightarrow$ hyperbolic representation matrix element $\langle s_1 | \, g \, | s_1 \rangle$ with hyperbolic label s_1 $\rightarrow$ cutoff!

Towards sine dilaton gravity

Quasi local energy $H=\sqrt{-2\cos(\Phi_b)}-\sqrt{-2\cos(\Phi_b)+2\cos(\theta)}$

$$\boxed{H_0=2\cos(\theta)=e^{-i\hat{\Pi}}+2e^{-i\Phi_b}e^{-\hat{\Lambda}}+\left(1-e^{-2i\Phi_b}e^{-\hat{\Lambda}}\right)e^{i\hat{\Pi}}\left(1-e^{-\hat{\Lambda}}\right)}$$

- H and H_0 are both real along an imaginary contour $\Phi_b\in[\pi,\pi+i\infty]$.
- As $\nu=i\Phi_b/\hbar\to\infty$, $\Lambda_{\mathrm{ren}}=\Lambda+2\nu$ it reduces to **DSSYK transfer matrix!**

Classical length Λ (after removing a pure complex phase $+i\pi/2$)

$$\Lambda=2\log\left(\sqrt{\frac{\left(e^{-ir_b}-\cos(\theta)\right)^2}{\sin^2(\theta)}+1}\cosh(\sin(\theta)t)\right)$$

Minimum value allowed in phase space at $\left(t=0,\theta=\arccos\left(e^{i\Phi_b}\right)\right)$:

$$\Lambda^{\mathrm{min}}=-2\nu\qquad\Lambda^{\mathrm{min}}_{\mathrm{ren}}=0$$

- explains the puzzle $\Lambda_{\mathrm{ren}}=2|\log q|n\geq 0$ **length positivity constraint!**
- Al-Salam-Chihara polynomials $Q_n\left(\cos(\theta)|e^{\nu},e^{\nu};q^2\right)$ is solution.

More on the choice of ordering

Using the mapping

$$\hat{\Lambda} \equiv 2\log\Big(\sinh\Big(\hat{L}/2\Big)\Big),$$
$$\hat{\Pi} \equiv \frac{1}{2}\Big(\hat{P}\,\tanh\Big(\hat{L}/2\Big) + \tanh\Big(\hat{L}/2\Big)\,\hat{P}\Big).$$

The $b \to 0$ limit of sinh dilaton is

$$\begin{aligned} &1 + \frac{B^2}{2}\hat{\Phi}_h^2 + O(B^4) \\ =&1 + \frac{B^2}{2}\left(\frac{1}{4\kappa^2}\hat{\Pi}^2(1+e^{-\hat{\Lambda}}) + \frac{1}{4\kappa^2}(1+e^{-\hat{\Lambda}})\hat{\Pi}^2 + \frac{1}{2\kappa^2}\hat{\Pi}(1+e^{-\hat{\Lambda}})\hat{\Pi} + \frac{\Phi_b^2}{1+e^{\hat{\Lambda}}}\right) \end{aligned}$$

The corresponding JT gravity prepotential operator takes the form

$$\begin{aligned} &\hat{\Phi}_h^2(\hat{L},\hat{P}) \\ &= \frac{3}{4}\hat{P}^2 + \frac{1}{8}\left(\frac{1}{\tanh(\hat{L}/2)}\hat{P}^2\tanh(\hat{L}/2) + \tanh(\hat{L}/2)\hat{P}^2\frac{1}{\tanh(\hat{L}/2)}\right) + \frac{\Phi_b^2}{\cosh^2(\hat{L}/2)}, \end{aligned}$$

which yields the JT Schroedinger equation and the reduced $sl(2,\mathbb{R})$ Casimir.

Flat ordering

$$Z(\beta) = \left(\lim_{L\to 0} L\right) \int_0^\infty \mathrm{d}s \, \frac{s \sinh(2\pi s)}{\pi^3} \Gamma\left(\frac{3}{4} \pm i\delta \pm is\right) e^{-\beta E(s)}.$$

where

$$s \equiv \frac{\Phi_h}{\hbar}, \qquad \frac{\Phi_b^2}{\hbar^2} \equiv \delta^2 + \frac{1}{16}.$$