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Equilibrium currents in anomalous fluid dynamics

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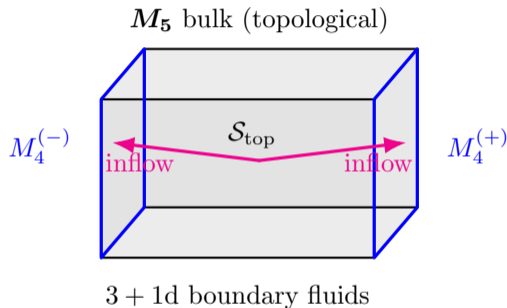
- **Scope.** The literature on anomalous hydrodynamics is extensive; here we focus on equilibrium phenomena and their variational origin.
- **Frame choice.** We work throughout in the **Landau frame** (as in [Son & Surowka \(2009\)](#)), while emphasizing an **action-based formulation**.
- **Effective action viewpoint.** We use **transgression forms** to encode anomalies in the effective action, following the approach of [Haehl, Loganayagam & Rangamani \(2014\)](#).
- **Entropy and symmetries.** The tension between anomalies and **chemical shift symmetry** was identified in [Dubovsky, Hui & Nicolis \(2014\)](#); here it appears through the structure of the entropy current.

Anomaly inflow: topological bulk + boundary fluid

- **5d topological insulator** on a slab M_5 .
- Boundary: two **3+1d hypersurfaces**:

$$\partial M_5 = M_4^{(+)} \sqcup M_4^{(-)}.$$

- We study **fluid dynamics on the boundary**.
- The **bulk** provides a source/sink for boundary charge via **anomaly inflow**.



Bulk is gapped/topological; boundary is dynamical.
Anomaly = boundary nonconservation balanced by 5d inflow.

Outline

- 1 Introduction: Hydrodynamic action
- 2 Fluid with chiral anomaly
- 3 Entropy current and chemical shift symmetry
- 4 Covariant current in equilibrium
- 5 Bloch theorem and magnetization current
- 6 Conclusions

Introduction: Hydrodynamic action

Thermodynamics

A fluid element is characterized by **enthalpy per particle** w and **entropy per particle** S . We treat the pressure and internal energy as

$$P = P(w, S), \quad \epsilon(n, nS).$$

From local thermodynamics,

$$dP = n d\mu + nS dT, \quad d\epsilon = \mu dn + T d(nS),$$

with n the particle density, T the temperature, and μ the chemical potential.

Using the identity

$$w = \mu + TS,$$

one obtains the equivalent form

$$\boxed{dP = n dw - nT dS.}$$

Fluid Action

The action of the fluid is taken as the spacetime integral of the pressure:

$$\mathcal{S}[\phi; A] = \int d^4x P(p_\mu, S),$$

with

$$p = \pi - A, \quad \pi = d\theta - S d\eta + \Phi_2 d\Psi_2 = d\theta + \Phi_\sigma d\Psi_\sigma, \quad \sigma = 1, 2,$$

and

$$\phi = \{\theta, S, \eta, \Phi_2, \Psi_2\} \quad (\text{Clebsch potentials}).$$

Here θ is the only field charged under the background $U(1)$ gauge field A_μ .

Redundancy (gauge freedom) of Clebsch form:

$$\theta \rightarrow \theta + f(\Psi), \quad \eta \rightarrow \eta, \quad \Phi \rightarrow \Phi - f'(\Psi),$$

which leaves π and \mathcal{S} invariant.

Thermodynamic identity: $dP = -n^\mu dp_\mu - nT dS$.

Equations of Motion

Varying the action $\mathcal{S}[\phi; A]$ with respect to the Clebsch fields, and using

$$n^\mu = -\frac{\partial P}{\partial p_\mu},$$

one obtains the basic equations:

$$\boxed{\partial_\mu n^\mu = 0} \quad (\text{particle conservation}), \quad \boxed{\partial_\mu (S n^\mu) = 0} \quad (\text{entropy conservation}),$$

together with

$$n^\mu \partial_\mu \eta = nT, \quad n^\mu \partial_\mu \Psi_\sigma = 0, \quad \partial_\mu (\Phi_\sigma n^\mu) = 0.$$

For a diffeomorphism variation $\delta\phi_a = \mathcal{L}_\xi \phi_a$, $\delta\pi_\nu = \mathcal{L}_\xi \pi_\nu = \xi^\mu \partial_\mu \pi_\nu + \pi_\mu \partial_\nu \xi^\mu$, the equations of motion imply

$$\boxed{n^\mu (\partial_\mu \pi_\nu - \partial_\nu \pi_\mu) = 0.} \quad (\text{Euler equation})$$

Energy–momentum conservation follows as another consequence:

$$\boxed{\partial_\mu T^\mu_\nu = F_{\nu\mu} n^\mu}, \quad T^\mu_\nu = n^\mu p_\nu + \delta^\mu_\nu P.$$

Remark: Covariance of the Formalism

The action depends on the momentum only through the scalar

$$P = P(w, S), \quad w = w(p_\mu), \quad p_\mu = \pi_\mu - A_\mu.$$

All equations of motion follow from variations of the Clebsch potentials and the identity

$$n^\mu = -\partial P / \partial p_\mu.$$

Choice of the scalar $w(p_\mu)$ determines the kinematics:

- Relativistic fluid:

$$w = \sqrt{-p_\mu p^\mu}, \quad p_\mu = w u_\mu, \quad u_\mu u^\mu = -1.$$

- Nonrelativistic (Galilean) fluid:

$$w = -p_0 - \frac{p_i p_i}{2}, \quad p_i = v_i, \quad n^0 = \rho.$$

The Clebsch–action formalism is **geometrically covariant**: the dynamics is fixed once we specify which scalar $w(p_\mu)$ encodes the spacetime symmetry.

Fluid with chiral anomaly

Right Weyl fermion in magnetic field: spectrum and degeneracy

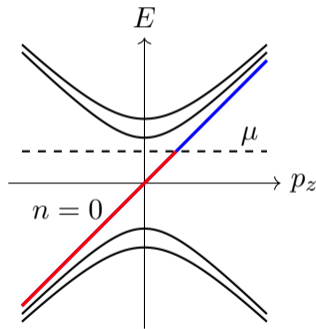
- Landau levels ($n \geq 1$):

$$E_n = \pm \sqrt{c^2 p_z^2 + 2eB \hbar c n}.$$

- **Zeroth Landau level (ZLL)** is chiral:

$$E_0(p_z) = +c p_z.$$

- Group velocity along B : $v_z = c$.
- Degeneracy per area: $\frac{1}{2\pi l_B^2} = \frac{eB}{2\pi \hbar c}$.
- Longitudinal DOS: $\frac{dp_z}{2\pi \hbar}$.



Chiral Magnetic Effect

At $T = 0$ the zeroth Landau level is filled for $0 \leq p_z \leq \mu/c$.

- Current along \mathbf{B} from the ZLL:

$$j_z = \int_0^{\mu/c} \left(\frac{dp_z}{2\pi\hbar} \right) \left(\frac{eB}{2\pi\hbar c} \right) (ev_z) = \frac{eB}{4\pi^2\hbar^2 c} \left(\frac{\mu}{c} \right) ec = \frac{e^2}{4\pi^2\hbar^2 c} \mu B.$$

- **Chiral Magnetic Effect** ($e = \hbar = c = 1$):

$$\boxed{j^{(R)} = \frac{1}{4\pi^2} \mu_R B}, \quad \boxed{j^{(L)} = -\frac{1}{4\pi^2} \mu_L B}.$$

- For a Dirac fermion:

$$\boxed{j = \frac{1}{2\pi^2} \mu_5 B} \quad (\text{CME}), \quad \boxed{j_5 = \frac{1}{2\pi^2} \mu B} \quad (\text{CSE})$$

Fluid with Chiral Anomaly: topological term

We add to the fluid action a topological (Wess–Zumino type) term:

$$\mathcal{S}[\phi; A] = \mathcal{S}_4 + \mathcal{S}_{\text{top}}, \quad \mathcal{S}_{\text{top}} = \alpha \int_{M_5} T_5(\pi, A),$$

where M_5 is a 5d manifold with boundary $\partial M_5 = M_4$ (physical spacetime).

The transgression 5-form is

$$T_5(\pi, A) = C_5(\pi) - C_5(A) + dB_4(\pi, A) \tag{1}$$

$$= \pi d\pi d\pi - A dA dA + d[A(\pi - A)d(\pi + A)] \tag{2}$$

$$= p \left(dp dp + 3 dp dA + 3 dA dA \right), \quad p = \pi - A. \tag{3}$$

- T_5 is **gauge invariant**.
- T_5 is **topological**: independent of the metric.
- Hence \mathcal{S}_{top} does *not* contribute to T^μ_ν .

Remark: Quantization of the Coefficient

The topological Wess–Zumino term

$$\mathcal{S}_{\text{top}} = \alpha \int_{M_5} T_5(\pi, A)$$

is defined using an extension of M_4 into a 5d manifold M_5 . Consistency requires the action to be independent of the choice of M_5 .

This fixes the overall coefficient to be **quantized**:

$$\alpha = \frac{k}{24\pi^2}, \quad k \in \mathbb{Z}.$$

- Quantization follows from $\int_{S^5} T_5 \in 48\pi^3\mathbb{Z}$, the standard normalization of the Abelian Chern–Simons 5-form.
- The choice $k = 1$ reproduces the **pure U(1) chiral gauge anomaly** of a single right-handed Weyl fermion:

Variation of the Topological Action

Variation of the topological (Wess-Zumino) term splits into a bulk anomaly term and boundary contributions:

$$\delta\mathcal{S}_{\text{top}} = \alpha \int_{M_5} \delta T_5(\pi, A) = \boxed{-3\alpha \int_{M_5} \delta A dA dA} \quad (\text{bulk anomaly})$$
$$+ \boxed{3\alpha \int_{M_4} (\delta\theta + \Phi_\sigma \delta\Psi_\sigma) d\pi d\pi} + \boxed{\alpha \int_{M_4} [\delta\pi p d(2p + 3A) + \delta A p d(p + 3A)]}$$

Resulting modification of currents:

$$\mathcal{J}^\mu = \mathcal{J}_{4d}^\mu - \alpha [p d(2p + 3A)]^\mu,$$
$$j^\mu = j_{4d}^\mu + \alpha [p d(p + 3A)]^\mu.$$

In the simplest case,

$$\mathcal{J}_{4d}^\mu = j_{4d}^\mu = n^\mu = -\frac{\partial P}{\partial p_\mu}.$$

Equations of Motion with Anomaly

Covariant current anomaly [from a gauge variation]

$$\partial_\mu j^\mu = -3\alpha [dA \wedge dA]$$

Euler (Carter–Lichnerowicz) equation [from a diffeo variation]

$$\mathcal{J}^\mu (\partial_\mu \pi_\nu - \partial_\nu \pi_\mu) = nT \partial_\nu S$$

Energy–momentum equation

$$\partial_\mu T^\mu_\nu = F_{\nu\mu} j^\mu$$

For a purely dynamical action $\mathcal{S}_4 = \int d^4x P(p_\mu, S)$, $T^\mu_\nu = n^\mu p_\nu + \delta^\mu_\nu P$

- The topological term does *not* contribute to T^μ_ν .
- It modifies the definition and conservation of the covariant current j^μ .

Entropy current and chemical shift symmetry

Entropy Current

The entropy equation $\mathcal{J}^\mu \partial_\mu S = 0$ does *not* have a conserved form. However, it can be rewritten as a conservation law:

$$\partial_\mu (S \hat{\mathcal{J}}^\mu) = 0$$

with the modified entropy current

$$\hat{\mathcal{J}}^\mu = \mathcal{J}^\mu + 2\alpha (\pi - d\theta) d\pi \tag{4}$$

$$= \mathcal{J}_{4d}^\mu - 2\alpha [(d\theta - A) d\pi]^\mu - \alpha [(\pi - A) dA]^\mu. \tag{5}$$

- The entropy current $s^\mu = S \hat{\mathcal{J}}^\mu$ is $U(1)$ **gauge invariant**.
- It **breaks chemical shift symmetry**: θ enters explicitly, not only via π .
c.f. Dubovsky, Hui, Nicolis, '14.

Covariant current in equilibrium

Chiral Magnetic and Chiral Vortical Effects

For a single **right-handed Weyl fermion** at constant chemical potential μ in equilibrium, in the presence of a uniform magnetic field \mathbf{B} and fluid vorticity

$$\boldsymbol{\omega} = \nabla \times \mathbf{p}, \quad \omega = dp,$$

the equilibrium current takes a simple form.

Equilibrium current:

Vilenkin (1980), ...

$$\mathbf{J}_{\text{eq}} = \frac{\mu}{8\pi^2} (\boldsymbol{\omega} + 2\mathbf{B})$$

For comparison, the general (consistent) currents in the effective theory are

$$\begin{aligned} \mathcal{J}^\mu &= n^\mu - \alpha [p d(2p + 3A)]^\mu, \\ j^\mu &= n^\mu + \alpha [p d(p + 3A)]^\mu. \end{aligned}$$

How does an equilibrium current emerge from these expressions?

Equilibrium Conditions

Assume time-independent gauge for gauge fields corresponding to stationary background

$$A_0(\mathbf{x}), \quad \mathbf{A}(\mathbf{x}).$$

- Construct the Hamiltonian corresponding to the effective action.
Monteiro, AGA, Nair '15
- Minimize the Hamiltonian with respect to the dynamical fields.
- Select the resulting stationary (equilibrium) configuration.
- Evaluate the covariant currents on this solution.

This procedure determines equilibrium (magnetization) currents in the presence of background fields.

Hamiltonian and Equilibrium Conditions

Perform the De Donder–Weyl (covariant Legendre) transform:

$$\mathcal{S} = \int d^4x \left[-n^\mu p_\mu + \epsilon(n^\mu, S) \right] + \alpha A \pi d(\pi + A) + \alpha \theta [d\pi d\pi].$$

Time derivatives of the Clebsch fields enter *linearly* and only through p_0 and the topological terms. The Hamiltonian therefore takes the form

$$\mathcal{H} = \int d^3x \left(n^0 A_0 + n^i p_i + \epsilon(n^\mu, S) \right) + \dots$$

Equilibrium is obtained by minimizing \mathcal{H} : **Wiegmann, AGA '22**

- Variation with respect to n^0 gives

$$A_0 + \frac{\partial \epsilon}{\partial n^0} = A_0 + p_0 = \pi_0 = 0, \quad \boxed{\pi_0 = 0}.$$

- Variation with respect to p_i gives vanishing particle current in equilibrium

$$\boxed{\mathcal{J}^i = 0}, \quad (\text{Beltrami flow})$$

Covariant Current in Equilibrium

The covariant and particle currents differ by anomaly-induced terms:

$$\begin{aligned}j^\mu - \mathcal{J}^\mu &= \alpha \left([p d(p + 3A)]^\mu + [p d(2p + 3A)]^\mu \right) \\ &= 3\alpha [p d(p + 2A)]^\mu.\end{aligned}\tag{6}$$

In equilibrium,

$$\mathcal{J}^i = 0, \quad \pi_0 = 0, \quad p_0 = -A_0 = -\mu.$$

Therefore the spatial covariant current is

$$\boxed{\mathbf{j}_{\text{cov}}^{\text{eq}} = 3\alpha \left[\mu(\boldsymbol{\omega} + 2\mathbf{B}) - \mathbf{p} \times \nabla\mu \right]}.\tag{7}$$

For uniform μ and $\alpha = \frac{1}{24\pi^2}$, this reproduces the standard **Chiral Magnetic** and **Chiral Vortical** equilibrium currents.

Bloch theorem and magnetization current

Bloch theorem and equilibrium currents

- **Bloch (no-current) theorem, thermodynamic form:**

For a U(1)-invariant many-body system with local interactions in static electromagnetic fields,

$$\lim_{V \rightarrow \infty} \frac{\langle \mathbf{J} \rangle_{\text{eq}}}{V} = 0.$$

There is **no macroscopic transport current density** in equilibrium.

- Finite-size systems (e.g. mesoscopic rings) may have persistent currents; the theorem constrains only the *thermodynamic-limit transport current density*.
- The theorem forbids net **transport** current, but allows local **magnetization** currents $\mathbf{J}_{\text{mag}} = \nabla \times \mathbf{M}$ with $\int d^3x \mathbf{J}_{\text{mag}} = 0$.
- In hydrodynamics this means: any “equilibrium CME/CVE current” must be a **magnetization current** (curl of a local quantity), not a true bulk transport current.

Bloch's argument (modern formulation after Yamamoto)

- Start from a U(1)-invariant many-body Hamiltonian H with local interactions on a torus. Apply a *uniform gauge twist* (parameter θ) through

$$H(\theta) = e^{i\theta X} H e^{-i\theta X},$$

where X generates a large gauge transformation.

- The equilibrium free energy $F(\theta) = -T \log \text{Tr} e^{-\beta H(\theta)}$ is periodic and minimized at some $\theta = \theta_*$.
- Differentiating at the minimum gives

$$0 = \left. \frac{\partial F}{\partial \theta} \right|_{\theta=\theta_*} = \langle J \rangle_{\text{eq}},$$

where J is the total U(1) current.

- A key estimate (using locality of interactions) shows that

$$\langle J \rangle_{\text{eq}} = \mathcal{O}(L^{d-1}),$$

Magnetization Currents and Hydrodynamics

Assume a fluid action without anomaly of the form

$$\mathcal{S} = \mathcal{S}[\pi - A, dA].$$

The particle and electric currents are defined as

$$\mathcal{J}^\mu = -\frac{\delta\mathcal{S}}{\delta p_\mu}, \quad j^\mu = \frac{\delta\mathcal{S}}{\delta A_\mu} = \frac{\delta\mathcal{S}}{\delta p_\mu} + 2\partial_\nu \frac{\delta\mathcal{S}}{\delta F_{\mu\nu}} = \mathcal{J}^\mu + J_{\text{mag}}^\mu.$$

The **magnetization current** is identically conserved

$$\boxed{J_{\text{mag}}^\mu = 2\partial_\nu \frac{\delta\mathcal{S}}{\delta F_{\mu\nu}}}, \quad \boxed{\partial_\mu J_{\text{mag}}^\mu = 0} \quad (\text{off-shell}).$$

In equilibrium, the particle current vanishes, $\mathcal{J}^i = 0$, and therefore

$$\boxed{j_{\text{eq}} = J_{\text{mag}}}.$$

Equilibrium currents in hydrodynamics are magnetization currents, not transport currents.

Magnetization Currents from the Topological Term

Consider the topological contribution to the fluid action

$$\mathcal{S}_{\text{top}} = \alpha \int_{M_5} T_5(\pi, A) = \alpha \int_{M_5} p \left(dp dp + 3 dp dA + 3 dA dA \right), \quad p = \pi - A.$$

Varying with respect to the gauge field A at fixed p (equivalently dA) yields

$$\delta \mathcal{S}_{\text{top}} = 3\alpha \int_{M_5} \delta A dp d(p + 2A) + 3\alpha \int_{M_4} \delta A p d(p + 2A).$$

The boundary term defines the associated **magnetization current**:

$$j_{\text{top, mag}}^\mu = 3\alpha [p d(p + 2A)]^\mu.$$

- This current coincides with the equilibrium current in a fluid with chiral anomaly.
- CME/CVE equilibrium currents arise from the **topological magnetization current**.

Conclusions

- **Chiral anomaly via inflow.** Adding the gauge-invariant transgression term $\mathcal{S}_{\text{top}} = \alpha \int_{M_5} T_5(\pi, A)$ yields the covariant anomaly, while leaving the energy–momentum tensor unchanged.
- **Entropy current.** An entropy current can still be written in conserved form, but the anomaly breaks the chemical shift symmetry.
- **Equilibrium current from the Hamiltonian.** Minimizing the Hamiltonian gives the covariant equilibrium current, reproducing Vilenkin’s combined CME/CVE expression for a single right-handed Weyl fermion.
- **Bloch theorem.** The equilibrium current is a *magnetization* current (not a transport current), consistent with the Bloch no-current theorem in the thermodynamic limit.