# Hydrodynamics with Anomalies and Effective Field Theory

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## Outline

- Intro: anomaly inflow
- Intro: 2d anomalies and bosonization of fermions
- 2d & 4d hydrodynamics with anomalies from inflow (perfect fluid)
- hydrodynamics from fermion path-integral

# Hydrodynamics: overview

- Effective low-energy description of fluid phases
- Like effective field theory, but written in terms of currents as bosonic "fields"

→ "non-particle" field theory (Jackiw, Nair, Pi, Polychronakos, 2004)

- Local equilibrium, local thermodynamics, based on symmetries only
- Many potential applications to strongly-interacting (fermionic) fluids
- Relation with gravity
- (Semi)classical, quantization is postponed; one-point functions and responses

→ Try to bridge with effective field theory, using recent advances

- hereafter: discuss hydrodynamics in Euler formulation

# Topological phases of matter & hydrodynamics

- Topological phases of matter provide new kinds of effective theories

  - BULK: gapped, with non-trivial global effects (Aharonov-Bohm phases)
    described by topological gauge theories (Chern-Simons theory, BF theory)

  - BOUNDARY: massless fermionic excitations & anomalies

  - anomaly inflow from bulk to boundary

→ Find here the relation with hydrodynamics

  - perfect fluid: T=0, dynamics from pressure & density $P = P(\rho)$

  - fluid action: variational formulation of Euler equations

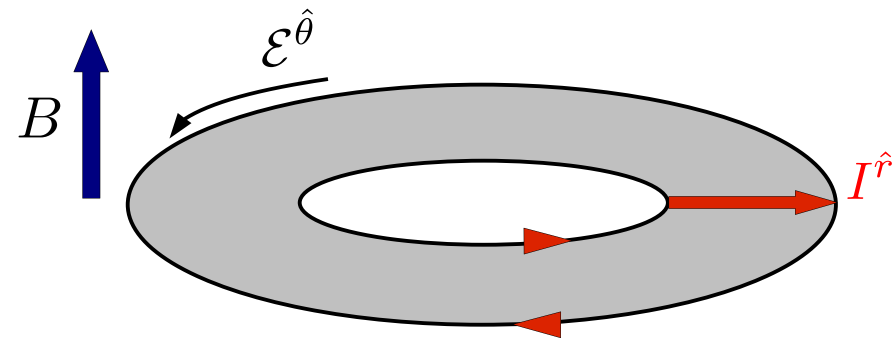
→ 2d: hydro is equal to standard bosonic QFT for fermions

→ 4d: can implement all anomalies in hydro, see universal/geometrical effects

→ derivation of hydro action from fermion path-integral

→ Euler hydrodynamics with/from anomalies

# Anomaly inflow




- quantum Hall effect, gapped 3d bulk, 2d massless chiral fermions on edges
- bulk Hall current described by Chern-Simons topological theory

$$S_{CS}[A] = -\frac{1}{4\pi}\int d^3x\ A\wedge dA, \qquad I^{\hat r} = -\frac{1}{2\pi}\varepsilon^{\hat r\alpha\beta}\partial_\alpha A_\beta, \qquad A = A_\mu dx^\mu, \quad \mu = 0,1,2,$$

- inflow: 2d anomaly compensated by classical current in one extra dimension

$$\oint d\theta\ I^{\hat r} = \frac{dQ_{edge}}{dt} = \oint d\theta\ \partial_\alpha J^\alpha, \qquad \partial_\alpha J^\alpha = -\frac{1}{2\pi}\varepsilon^{\alpha\beta}\partial_\alpha A_\beta, \quad \alpha,\beta = 0,1$$

$J^\alpha$ 2d current
$I^\mu$ 3d current

- Charge is conserved in bulk+boundary system

# anomaly inflow and bulk-boundary fields

- Topological insulators (Dirac fermions on boundary): BF theory, `hydrodynamic' gauge fields $p_\mu, \tilde{p}_\mu$ express bulk matter currents, $I^\mu = \varepsilon^{\mu\nu\rho}\partial_\nu \tilde{p}_\rho, \quad \tilde{I}^\mu = \varepsilon^{\mu\nu\rho}\partial_\nu p_\rho$

$$S_{BF} = \int \tilde{p}dp + \tilde{p}dA + \tilde{A}dp = \int \underbrace{(\tilde{p}+\tilde{A})d(p+A)}_{\tilde{\pi}d\pi} - \tilde{A}dA \underset{eom}{=} -\int \tilde{A}dA \qquad (e/\pi = 1)$$

- equations of motion

$$d\pi = dp + dA = 0 \quad \rightarrow \quad p \underset{eom}{=} d\theta - A$$

$$d\tilde{\pi} = d\tilde{p} + d\tilde{A} = 0 \quad \rightarrow \quad \tilde{p} \underset{eom}{=} d\psi - \tilde{A}$$

- anomaly inflow

$$\tilde{I}^{\hat{r}} = \varepsilon^{\hat{r}\alpha\beta}\partial_\alpha p_\beta = \partial_\alpha \tilde{J}^\alpha \ \rightarrow \ \tilde{J}^\alpha = \varepsilon^{\alpha\beta}\partial_\alpha p_\beta$$

$J^\alpha$ 2d current
$I^\mu$ 3d current

→ hydrodynamic fields $p, \tilde{p}$ express the boundary currents, once reduced to 2d

$$\tilde{J}^\alpha = \varepsilon^{\alpha\beta} p_\beta \underset{eom}{=} \varepsilon^{\alpha\beta}(\partial_\beta\theta - A_\beta) \ \leftarrow$$

$$J^\alpha = \varepsilon^{\alpha\beta}\tilde{p}_\beta \underset{eom}{=} \varepsilon^{\alpha\beta}(\partial_\beta\psi - \tilde{A}_\beta) \ \leftarrow$$

$p, \tilde{p}$ gauge inv. on boundary

→ spoiler: these fields express bosonization and Euler hydrodynamics

$$p = \pi - A, \quad \tilde{p} = \tilde{\pi} - \tilde{A}, \qquad d\pi = d\tilde{\pi} = 0$$

# 2d bosonization and anomalies

- many derivations of bosonization, non-perturbative, endless number of applications

$$S = \frac{1}{2}\int d^2x \; (\partial_\mu\theta)^2 \,, \qquad \theta(t,x)$$

- U(1) symmetry $\theta \to \theta + \mathrm{const.}$ but two conserved currents

$$J^\mu = \partial_\mu\theta, \qquad \partial_\mu J^\mu = 0 \qquad \text{Noether current}$$

$$\tilde{J}^\mu = \varepsilon^{\mu\nu}\partial_\nu\theta, \qquad \partial_\mu \tilde{J}^\mu = 0 \qquad \text{topological (axial) current}$$

- like Dirac fermion: $J^\mu = \bar{\psi}\gamma^\mu\psi, \qquad \tilde{J}^\mu = \bar{\psi}\gamma^5\gamma^\mu\psi$

- couple to corresponding backgrounds $A_\mu, \tilde{A}_\mu$

$$S = \int d^2x \frac{1}{2}\left(\partial_\mu\theta - A_\mu\right)^2 + \tilde{A}_\mu\varepsilon^{\mu\nu}\left(\partial_\nu\theta - A_\nu\right) - \int_{\mathcal{M}_3}\tilde{A}dA, \qquad \text{(for g.i.)}$$

- gauge-invariant currents ("covariant currents") are anomalous

$$J^\mu = \partial_\mu\theta - A_\mu, \qquad \partial_\mu J^\mu = -\varepsilon^{\mu\nu}\partial_\mu\tilde{A}_\nu, \qquad (e/\pi \to 1)$$

$$\tilde{J}^\mu = \varepsilon^{\mu\nu}\left(\partial_\nu\theta - A_\nu\right), \qquad \partial_\mu\tilde{J}^\mu = -\varepsilon^{\mu\nu}\partial_\mu A_\nu$$

→ same currents of inflow $J^\mu = \partial_\mu\theta - A_\mu = \varepsilon^{\mu\nu}(\partial_\nu\psi - \tilde{A}_\nu)$ duality $(\theta, A) \leftrightarrow (\psi, \tilde{A})$

# Action of perfect fluid

- long history (Lichnerowicz; Carter; Arnold; Marsden;......; Abanov, Wiegmann '22)
- action of fluid momentum $p_\mu$ (T=0, s=const.)

$$S = -\int \mathcal{J}^\alpha p_\alpha + E(\mathcal{J}) \;\rightarrow\; S[p] = \int P(p),$$

$$p_\alpha = -\partial E/\partial \mathcal{J}_\alpha \qquad dP = \rho\, d\mu, \quad \mu = \mu(p_\alpha)$$

| | |
|---|---|
| $P$ | pressure |
| $\mu$ | chemical potential |
| $\rho$ | fluid density |
| $p_\alpha = \mu u_\alpha,$ | $u^2 = -1$ |

- Euler hydrodynamics is a constrained system
- Diffeo variations of S[p] or Clebsches give the Carter-Lichnerowicz equations

$$\mathcal{J}^\nu(\partial_\nu p_\mu - \partial_\mu p_\nu) = 0, \qquad \mathcal{J}^\nu = -\frac{\delta S}{\delta p_\nu}, \quad \partial_\mu \mathcal{J}^\mu = 0$$

→ solution in 2d: $\partial_\nu p_\mu - \partial_\mu p_\nu = 0 \;\rightarrow\; p_\mu \underset{eom}{=} \partial_\mu \theta$ $\qquad$ $\mathcal{J}$-independent = constraint

→ conserved axial current $\tilde{J}^\mu = \varepsilon^{\mu\nu} p_\nu$

- couple to backgrounds

$$S = \int_{\mathcal{M}_2} P(\pi - A) + \tilde{A}(\pi - A) \qquad p = \pi - A$$

- CL equation: $d\pi = 0$ $\quad$ axial current $\tilde{J}^\mu = \varepsilon^{\mu\nu}(\pi_\nu - A_\nu) \underset{eom}{=} \varepsilon^{\mu\nu}(\partial_\nu\theta - A_\nu)$

→ same fields, eom, anomalies, as bosonic theory and inflow from bulk BF theory !!

# 4d perfect fluids with anomalies

$$S[p] = \int d^4x\, P(p)$$

- CL equations of motion: $i_{\mathcal{J}} dp = 0 \quad \to \quad i_{\mathcal{J}}(dpdp) = 0 \quad \to \quad dpdp = 0$

→ solution is again a constraint

→ defines again the conserved axial current: $\tilde{J}_\mu$ helicity current

$$\tilde{J} = *pdp, \qquad \tilde{J}^\mu = \varepsilon^{\mu\nu\rho\sigma} p_\nu \partial_\rho p_\sigma, \qquad \partial_\mu \tilde{J}^\mu = 0$$

$$\tilde{Q} = \int d^3x\, \tilde{J}^0 \sim \int \vec{v}\cdot\vec{\omega}, \qquad \vec{\omega} = \nabla \times \vec{v}$$

- identify it as the axial current! (Son, Surowka '09; ........)
- write action with axial coupling (Abanov, Wiegmann '22)

$$S[\pi, A, \tilde{A}] = \int_{\mathcal{M}_4} P(\pi - A) + \tilde{A}(\pi - A)d(\pi + A) \qquad \tilde{J} = *pd(p + 2A), \quad p = \pi - A$$

→ obtain the anomalies of 4d Dirac fermions, but one term is missing...

$$d * \tilde{J} = -dAdA - d\tilde{A}d\tilde{A}, \qquad d * J = -2dAd\tilde{A}, \qquad \left(\frac{e}{2\pi} \to 1\right)$$

# 4d hydrodynamics from inflow

- 4+1d topological theory leading to 4d anomalies (Dirac fermion $\alpha = \frac{1}{3}$)

$$S_5 = \int_{\mathcal{M}_5} \tilde{\pi} d\pi d\pi + \alpha\, \tilde{\pi} d\tilde{\pi} d\tilde{\pi} - \tilde{A} dA dA + \alpha \tilde{A} d\tilde{A} d\tilde{A} \underset{eom}{\rightarrow} - \int_{\mathcal{M}_5} \tilde{A} dA dA + \alpha \tilde{A} d\tilde{A} d\tilde{A}$$

- write g.i. 4d fluid action with backgrounds and 4+1d terms for inflow

$$S = \int_{\mathcal{M}_4} P(\pi - A, \tilde{\pi} - \tilde{A}) + \tilde{A}(\pi - A) d(\pi + A) + \alpha \tilde{A}(\tilde{\pi} - \tilde{A}) d(\tilde{\pi} - \tilde{A}) + S_5$$

→ two fluid variables: $p = \pi - A, \quad \tilde{p} = \tilde{\pi} - \tilde{A}$ too many in general

→ reduction to one-fluid theory: $\tilde{\pi} = d\psi, \quad P = P(\pi - A)$ (minimal choice)

$$p = \pi - A, \qquad \tilde{p} = d\psi - \tilde{A}, \qquad \text{(gauge inv.)}$$

$$S[\pi, \psi] = \int_{\mathcal{M}_4} P(\pi - A) + \tilde{A}(\pi - A) d(\pi + A) + \psi \left(d\pi d\pi + \frac{1}{3} d\tilde{A} d\tilde{A}\right) - \int_{\mathcal{M}_5} \tilde{A} dA dA + \frac{1}{3} \tilde{A} d\tilde{A} d\tilde{A}$$

→ extends earlier hydro and yields correct anomalies

# 4d hydrodynamics from inflow

$$S[\pi,\psi]=\int_{\mathcal{M}_4} P(\pi-A)+\tilde{A}(\pi-A)d(\pi+A)+\psi(d\pi d\pi+\frac{1}{3}d\tilde{A}d\tilde{A})-\int_{\mathcal{M}_5}\tilde{A}dAdA+\frac{1}{3}\tilde{A}d\tilde{A}d\tilde{A}$$

$$p=\pi-A,\quad \tilde{p}=d\psi-\tilde{A}$$

Results

- $\psi$ is the missing pseudoscalar WZW field; Lagrange multiplier for CL equation.

- can also give dynamics to it $P=P(\pi-A,d\psi-\tilde{A})$

- interprete $\tilde{A}_0-\dot{\psi}$ ~ dynamical axial chemical potential, can model chiral breaking

- hydro provides an effective 4d bosonic theory for interacting fermions $P(p_\alpha)$

- "geometric" description of anomalies, independent of specific dynamics

- can also describe the 4d axial-gravitational anomaly

- NEXT: derivation from fermion path-integral

- Related works: Jensen, Loganayagam, Yarom '13; Haehl, Loganayagam, Rangamani '14,…

  (anomaly inflow, shadow field, transgression (see later))

# Path-integral derivation: chiral fermion

- free fermion $Z[A] = \int \mathcal{D}\chi\mathcal{D}\bar{\chi}\exp iS_F[\chi, A], \qquad S_F = \int_{\mathcal{M}_4} \bar{\chi}\gamma^\mu(i\partial_\mu + A_\mu)\chi$
- anomaly $$Z[A+d\lambda] = Z[A]\exp(i\alpha\int_{\mathcal{M}_4}\lambda dAdA), \qquad \alpha = \frac{1}{6}, \quad \left(\frac{e}{2\pi}\to 1\right)$$
- cancellation by inflow $Z[A] \to Z[A]\exp(-i\alpha\int_{\mathcal{M}_5} AdAdA), \qquad Z[A+d\lambda] = Z[A]$
- assume interactions lead to IR fluid phase: there remains an irrelevant current-current interaction $S_{int} = \int -\varepsilon(J) \sim -\frac{\sigma}{M^2}\int J_\mu J^\mu$
- Introduce Lagrange multiplier $p_\mu$ to obtain an effective action for current

$$Z[A] = \int \mathcal{D}\chi\mathcal{D}\bar{\chi}\mathcal{D}p\mathcal{D}J\exp(i\int \bar{\chi}\gamma^\mu(i\partial_\mu + A_\mu)\chi + p_\mu(\bar{\chi}\gamma^\mu\chi - J^\mu) - \varepsilon(J) - i\alpha\int_{\mathcal{M}_5} AdAdA)$$

➡ integrate $J_\mu$ by Legendre $-\varepsilon(J) - p^\mu J_\mu = P(p), \qquad J^\mu = -\partial P/\partial p_\mu, \qquad p_\mu = -\partial\varepsilon/\partial J^\mu$

➡ integrate fermions: background shift $A_\mu \to \pi_\mu = p_\mu + A_\mu,$

➡ $$Z[A] = \int \mathcal{D}p\exp(i\int_{\mathcal{M}_4} P(p) + \text{local terms} + i\alpha\int_{\mathcal{M}_5}\pi d\pi d\pi - AdAdA)$$

$$Z[A] = \int \mathcal{D}p \exp(i \int_{\mathcal{M}_4} P(p) + \underbrace{\text{local terms} + \alpha \int_{\mathcal{M}_5} \pi d\pi d\pi - AdAdA}_{T_5[\pi, A]}), \qquad (p = \pi - A, \text{ g.i.})$$

- $Z[A]$ still gauge invariant due to anomaly matching. Then $T_5(\pi, A)$ is g.i., it is the transgression, known quantity of anomaly descent equations from 6d:

  - take $F^3, \quad F = dA: \quad dF^3 = \delta_\lambda F^3 = 0$ then $F^3_A = dC_5(A)$ locally, Chern-Simons

  - since $p = \pi - A$ is g. i. vector, $\pi, A$ transform in same way & same cohomology

  -then $F^3_\pi - F^3_A = dT_5(\pi, A), \quad T_5$ is globally defined, g.i., unique, found by integration

$$S_{eff} = \int_{\mathcal{M}_4} P(\pi - A) + \overbrace{\alpha A(\pi - A)d(\pi + A) + \alpha \int_{M_5} \pi d\pi d\pi - AdAdA}^{T_5(\pi, a)}$$

→ obtained hydro action in chiral case; derivation extends to Dirac case

→ topological bulk eom $d\pi d\pi = 0$ (brought to 4d, diff. inv.) matches CL equation

→ Euler hydro with/from anomalies

# Conclusions

- Topological phases of matter: new view on anomalies, new effective field theories

➡ anomaly inflow from d+1 determines Euler hydrodynamics/bosonic effective theory

➡ d+1 topological action is needed; variational formulations match

➡ independent derivation by path-integral. In Dirac case:

- V-V interaction gives one-fluid theory; V-V & A-A interactions, the two-fluid theory

- both involve additional gauge-invariant terms with 1-3 free coefficients

## Generalizations/Ongoing

- universal, model-independent equilibrium currents: CME, CVE, CSE (cf. Abanov talk)
- add temperature and entropy; extend to many species & non-Abelian symmetries
- further 5d topological theories with 2-, 3-forms suggest other hydrodynamics, involving generalized symmetries; e.g. $\tilde{J} = *pdp \rightarrow *db$ independent 2-form