Event Shapes at Finite Density

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[Based on 2503.21867 G. Cuomo, E. Firat, L. Ricci, FN]

Outline

- Introduction and Motivations
- Event Shapes of Heavy States
- Event Shapes of Large Charge States
- Conclusions and Future Directions

Introduction and Motivations

Energy detector [Sterman 1975]

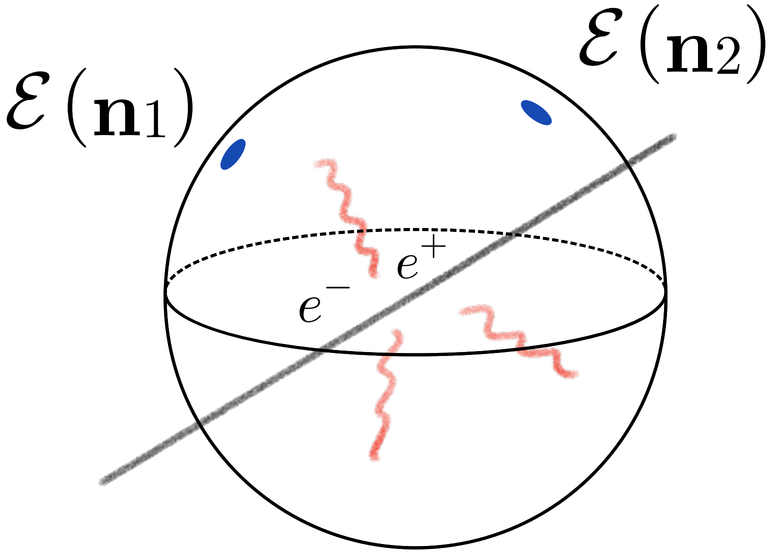
$$\mathcal{E}(\mathbf{n}) = \lim_{r\to\infty} r^2 \int_0^\infty dt\, \mathbf{n}_i\, T^{0i}(t,r,\mathbf{n})$$

- Can be computed theoretically and measured experimentally

1-pt event shape $\langle e^+e^-|\,\mathcal{E}(\mathbf{n})\,|e^+e^-\rangle$

n-pt event shape $\langle e^+e^-|\,\mathcal{E}(\mathbf{n}_1)...\mathcal{E}(\mathbf{n}_n)\,|e^+e^-\rangle$

[Basham Brown Ellis Love 1978]



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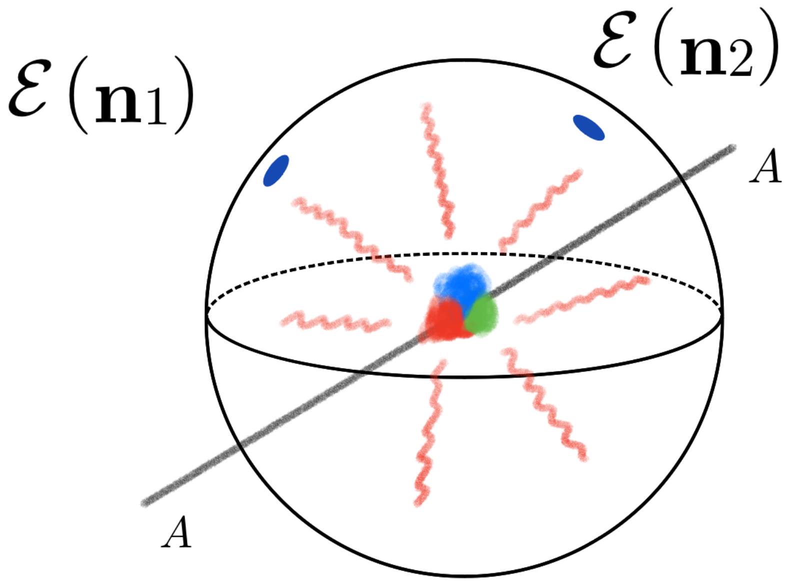

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[Basham Brown Ellis Love 1978]

- Can be used to study highly excited states of matter in colliders
[Andres Dominguez Elayavalli Holguin Marquet Moult 2022, Review: Moult Zhu 2025]

Conformal Collider

[Hofman Maldacena 2008, Hartman et al. 2018, Simon Caron-Huot et al. 2022 ...]

- Mostly studied in states close to the vacuum
- What are the universal properties of event shapes of highly excited states?

Event Shapes of Heavy States

Event shapes in CFT

$$\mathcal{D}_{\mathcal{O}}(\mathbf{n}) = 2^{-\Delta} \lim_{r^+\to\infty} (r^+)^{\Delta-J} \int_{-\infty}^{\infty} dr^-\, \mathcal{O}^{(\Delta,J)}_{\underbrace{-\cdots-}_{J}}(r^+, r^-, \mathbf{n})$$

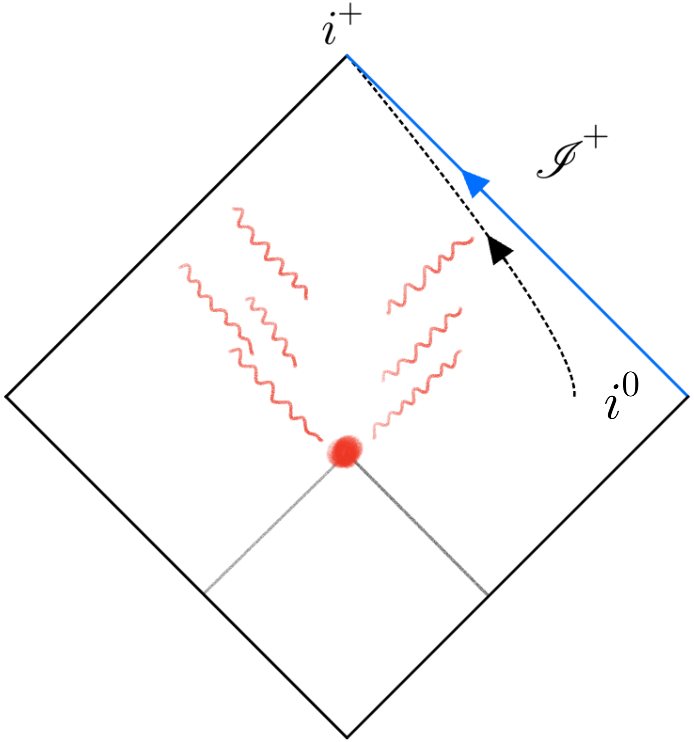
Excite the vacuum

$$\mathcal{O}_Q(p)\,|0\rangle = \int d^dx e^{-ip\cdot x}\, \mathcal{O}_Q(x)\,|0\rangle$$

and take a measurement

$$\langle \mathcal{D}_1(\mathbf{n}_1)...\mathcal{D}_n(\mathbf{n}_n)\rangle_p \equiv \frac{\langle \bar{\mathcal{O}}_Q(p)\mathcal{D}_1(\mathbf{n}_1)...\mathcal{D}_n(\mathbf{n}_n)\mathcal{O}_Q(p)\rangle}{\langle \bar{\mathcal{O}}_Q(p)\mathcal{O}_Q(p)\rangle}$$

1-pt event shape $\qquad \langle \mathcal{E}(\mathbf{n})\rangle_p = \dfrac{E}{\Omega_{d-2}}$



Event Shapes of Heavy States

- $\mathcal{O}_Q$, charged scalar, $\Delta_Q \gg 1$,

$$\langle \mathcal{D}_1(\mathbf{n}_1)\mathcal{D}_2(\mathbf{n}_2)\rangle_p = \frac{\int d^d z e^{ip\cdot z}\langle \bar{\mathcal{O}}_Q(z/2)\mathcal{D}_1(\mathbf{n}_1)\mathcal{D}_2(\mathbf{n}_2)\mathcal{O}_Q(-z/2)\rangle}{\int d^d z\, e^{ip\cdot z}\langle \bar{\mathcal{O}}_Q(z/2)\mathcal{O}_Q(-z/2)\rangle}$$

$$\langle \bar{\mathcal{O}}_Q(z/2)\,\mathcal{D}_1(\mathbf{n}_1)\mathcal{D}_2(\mathbf{n}_2)\,\mathcal{O}_Q(-z/2)\rangle = \frac{1}{(-z^2)^{\Delta_Q}}\frac{\langle \bar{\mathcal{O}}(z/2)\mathcal{D}_1(\mathbf{n}_1)\mathcal{D}_2(\mathbf{n}_2)\mathcal{O}(-z/2)\rangle}{\langle \bar{\mathcal{O}}(z/2)\mathcal{O}(-z/2)\rangle}$$

Event shapes of Heavy States

$$\langle \mathcal{D}_1(\mathbf{n}_1)\mathcal{D}_2(\mathbf{n}_2)\rangle_p \propto \int d^d z \frac{e^{ip\cdot z}}{(-z^2)^{\Delta_Q}} \frac{\langle \bar{\mathcal{O}}(z/2)\mathcal{D}_1(\mathbf{n}_1)\mathcal{D}_2(\mathbf{n}_2)\mathcal{O}(-z/2)\rangle}{\langle \bar{\mathcal{O}}(z/2)\mathcal{O}(-z/2)\rangle}$$

- Compute the integral via saddle point approximation $p^\mu = (E, \mathbf{0})$

$$z^\mu_{\text{Saddle}} = -i\left(X = \frac{2\Delta_Q}{E}, \mathbf{0}\right)$$

$$\langle \mathcal{D}_1(\mathbf{n}_1)\mathcal{D}_2(\mathbf{n}_2)\rangle_p = \frac{\langle \bar{\mathcal{O}}(-iX/2)\mathcal{D}_1(\mathbf{n}_1)\mathcal{D}_2(\mathbf{n}_2)\mathcal{O}(iX/2)\rangle}{\langle \bar{\mathcal{O}}(-iX/2)\mathcal{O}(iX/2)\rangle}\left(1 + \mathcal{O}\left(\frac{1}{\Delta_Q}\right)\right)$$

- Corrections computed with insertions of momentum operator

$$\mathcal{O}_Q\left(\frac{i}{2}X - \delta z\right) = \mathcal{O}_Q\left(\frac{i}{2}X\right) - i\delta z^\mu \left[P_\mu, \mathcal{O}_Q\left(\frac{i}{2}X\right)\right] + \dots .$$

Event shapes of Heavy States

$$\langle \mathcal{D}_1(\mathbf{n}_1)\mathcal{D}_2(\mathbf{n}_2)\rangle_p = \frac{\langle \bar{\mathcal{O}}(-iX/2)\mathcal{D}_1(\mathbf{n}_1)\mathcal{D}_2(\mathbf{n}_2)\mathcal{O}(iX/2)\rangle}{\langle \bar{\mathcal{O}}(-iX/2)\mathcal{O}(iX/2)\rangle}\left(1+\mathcal{O}\left(\frac{1}{\Delta_Q}\right)\right)$$

- Further semiclassical expansion

$$\langle \mathcal{D}(\mathbf{n}_1)\mathcal{D}(\mathbf{n}_2)\rangle_p = \langle \mathcal{D}(\mathbf{n}_1)\mathcal{D}(\mathbf{n}_2)\rangle^{(0)} + \frac{1}{\Delta_Q}\langle \mathcal{D}(\mathbf{n}_1)\mathcal{D}(\mathbf{n}_2)\rangle^{(1)} + \dots\,,$$

- We need to compute

$$\langle \bar{\mathcal{O}}_Q(-iX/2)[...]\mathcal{O}_Q(iX/2)\rangle$$

Event shapes of Heavy States

$$\langle J^\mu(x)\rangle_X = n_X(x)\, u_X^\mu(x), \qquad \langle T^{\mu\nu}(x)\rangle_X = \frac{d}{d-1}\rho_X(x)\left(u_X^\mu(x)u_X^\nu(x) - \frac{1}{d}\eta^{\mu\nu}\right),$$

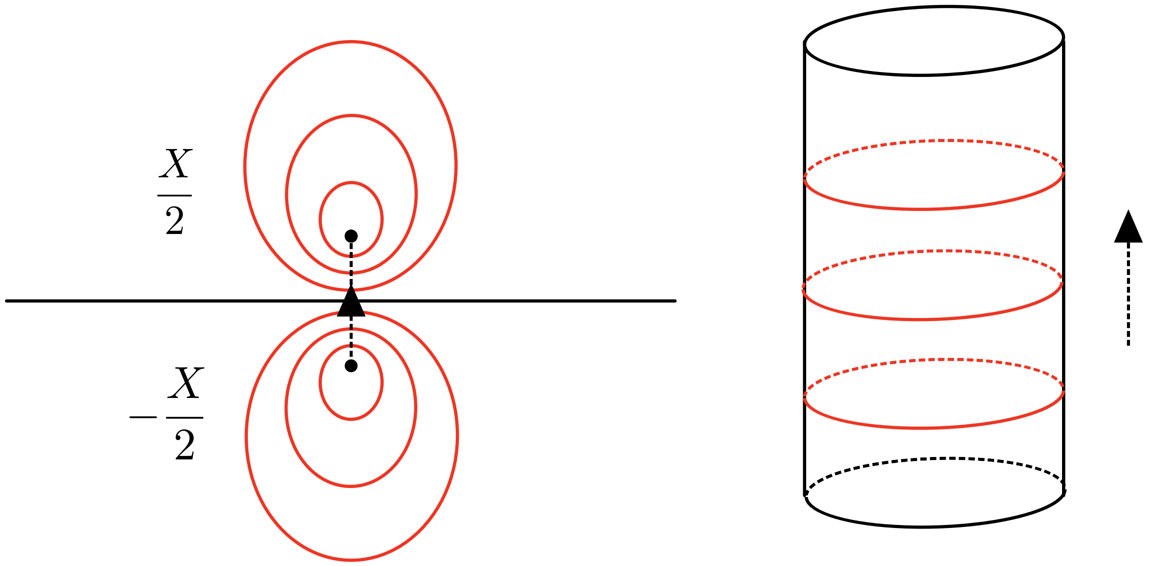
(a) $x^0 = 0$ (b) $x^0 = 2X$ (c) $x^0 = 4X$

Figure: Charge density in $d = 3$ $\qquad X = \frac{2\Delta}{E}$

- The state has a natural interpretation in the NS quantization of the theory

$$(X)^{\Delta}\,\mathcal{O}_Q(iX/2)\,|0\rangle = |\mathcal{O}_Q\rangle\,, \qquad D_X = \frac{X}{4}P^0 + \frac{1}{X}K^0, \qquad D_X\,|\mathcal{O}_Q\rangle = \Delta_Q\,|\mathcal{O}_Q\rangle\,.$$

Euclidean Plane-Cylinder map



Lorentzian Plane-Cylinder map
[Lüscher Mack 1975]

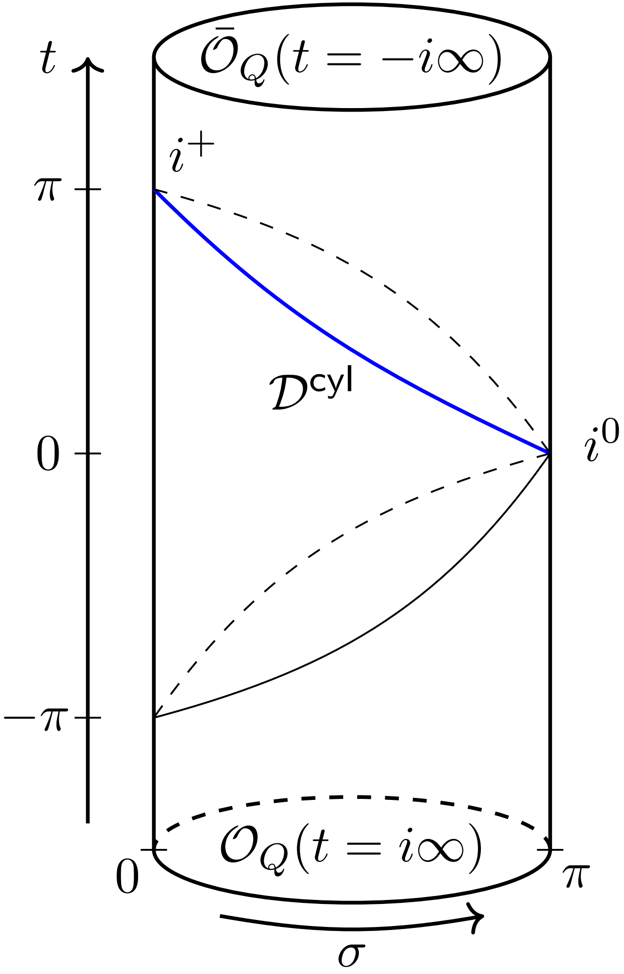
$$x^\mu = \frac{X}{2}\left(\frac{\sin t}{\cos t + \cos\sigma}, \frac{\sin\sigma}{\cos t + \cos\sigma}\mathbf{n}\right)$$

$t^\pm = t \pm \sigma,$ Future null infinity: $t^+ = \pi$

$$ds^2_{\mathsf{Cyl}}|_{\mathscr{I}^+} = -\cos^2\left(\frac{t^-}{2}\right) d^2\Omega_{S^{d-2}}(\mathbf{n})$$

- Detector on the cylinder

$$\mathcal{D}^{\mathsf{cyl}}(\mathbf{n}) \propto \int_{-\pi}^{\pi} dt^- \left(\cos\frac{t^-}{2}\right)^{\Delta+J-2} \mathcal{O}^{\mathsf{Cyl}}_{-\dots-}(t^+ = \pi, t^-, \mathbf{n})$$



Event shapes of Heavy States

- From Fourier transform to Cylinder correlators

$$\langle \mathcal{D}_1(\mathbf{n}_1)\mathcal{D}_2(\mathbf{n}_2)\rangle_p = \prod_{i=1,2}\left(\frac{2\Delta}{E}\right)^{1-J_i} \langle \mathcal{O}_Q|\, \mathcal{D}_1^{\text{Cyl}}(\mathbf{n}_1)\mathcal{D}_2^{\text{Cyl}}(\mathbf{n}_2)\,|\mathcal{O}_Q\rangle \left(1+\mathcal{O}\left(\frac{1}{\Delta_Q}\right)\right)$$

- Cylinder correlators from Semiclassics
  - $Q \gg 1$: Conformal Superfluid

    [Hellerman et al. 2015, Monin et al. 2016, Cuomo 2020]
  - $\Delta_Q \gg 1$: Finite Temperature Conformal Fluid/Superfluid

    [Delacretaz 2020]

Event Shapes of Large Charge States

Large Charge States: Conformal Superfluid

$$H \times U(1) \to H - \mu Q$$

- Realized by Goldstone boson χ with background value $\chi = \chi_0 + \mu t$ [Son 2001]

$$S[\chi] = c \int d^d x \sqrt{g} \left(\sqrt{g^{ab} \partial_a \chi \, \partial_b \chi} \right)^d \left(1 + \mathcal{O}(\partial^2/\Lambda^2) \right)$$

- Higher derivative corrections suppressed by gap $\Lambda \propto \mu = \mathcal{O}\left(Q^{\frac{1}{d-1}} \right)$.

Example: Scaling dimension of the large charge primary $\mathcal{O}_Q$

$$S[\chi] = c\int d^dx\sqrt{g}\left(\sqrt{g^{ab}\partial_a\chi\,\partial_b\chi}\right)^d + \ldots$$

$$J_a^{\mathsf{Cyl}} = c\,\partial_a\chi(\partial\chi)^{d-2} \qquad T_{ab}^{\mathsf{Cyl}} = c\left[\partial_a\chi\partial_b\chi(\partial\chi)^{d-2} - \frac{1}{d}g_{ab}(\partial\chi)^d\right]$$

$$Q = \int d\Omega_{S^{d-1}}\,\langle\mathcal{O}_Q|\,J_0^{\mathsf{Cyl}}\,|\mathcal{O}_Q\rangle \propto \mu^{d-1} \qquad \Delta_Q = \int d\Omega_{S^{d-1}}\,\langle\mathcal{O}_Q|\,T_{00}^{\mathsf{Cyl}}\,|\mathcal{O}_Q\rangle \propto \mu^d$$

$$\mu \propto Q^{\frac{1}{d-1}} \qquad \Delta_Q \propto Q^{\frac{d}{d-1}} + \ldots$$

- Quadratic action for the fluctuations $\chi = \chi_0 + \mu t + \pi$,

$$S^{(2)}[\pi] \propto c\mu^{d-2} \int dt\, d\Omega_{S^{d-1}} \left(\frac{1}{2}(\partial_t \pi)^2 - \frac{v_s^2}{2}(\partial_i \pi)^2 \right), \qquad v_s^2 = \frac{1}{d-1},$$

$$G_{\pi\pi}(t,x) = -i\frac{t}{2\Omega_{d-1}} + \sum_{l=1}^{\infty} \frac{2l+d-2}{(d-2)\Omega_{d-1}} \frac{e^{-i\omega_l t}}{2\omega_l} C_l^{\frac{d-2}{2}}(x), \qquad \omega_l = v_s\sqrt{l(l+d-2)},$$

- Stress tensor 2-pt function

$$\langle \mathcal{O}_Q | T^{\text{Cyl}}\, T^{\text{Cyl}} | \mathcal{O}_Q \rangle = \langle \mathcal{O}_Q | T^{\text{Cyl}} | \mathcal{O}_Q \rangle \langle \mathcal{O}_Q | T^{\text{Cyl}} | \mathcal{O}_Q \rangle \left(1 + \frac{\#}{\Delta_Q}\, \partial^2 G_{\pi\pi} + \ldots \right)$$

Event Shapes of Large Charge States: Leading Order Factorization

$$\langle \mathcal{E}(\mathbf{n}_1)\mathcal{E}(\mathbf{n}_2)\rangle_p = \prod_{i=1,2}\left(\frac{2\Delta}{E}\right)^{1-J_i} \langle \mathcal{O}_Q|\, \mathcal{E}^{\text{Cyl}}(\mathbf{n}_1)\mathcal{E}^{\text{Cyl}}(\mathbf{n}_2)\, |\mathcal{O}_Q\rangle \, \left(1+\mathcal{O}\left(\frac{1}{\Delta_Q}\right)\right)$$

$$\langle \mathcal{O}_Q|\, T^{\text{Cyl}}_{--}(t_1^-,\mathbf{n})T^{\text{Cyl}}_{--}(t_1^-,\mathbf{n})\, |\mathcal{O}_Q\rangle = \langle \mathcal{O}_Q|T^{\text{Cyl}}_{--}|\mathcal{O}_Q\rangle\langle \mathcal{O}_Q|T^{\text{Cyl}}_{--}|\mathcal{O}_Q\rangle\left(1+\mathcal{O}\left(\frac{1}{\Delta_Q}\right)\right)$$

$$\langle \mathcal{E}(\mathbf{n}_1)\mathcal{E}(\mathbf{n}_2)\rangle^{(0)} = \langle \mathcal{E}(\mathbf{n}_1)\rangle\langle \mathcal{E}(\mathbf{n}_2)\rangle = \left(\frac{E}{\Omega_{d-2}}\right)^2$$

Universal feature of heavy states

[Chicherin Korchemsky Sokatchev Zhiboedov 2023, Firat Monin Rattazzi Walters 2023]

Event Shapes of Large Charge States: Subleading Order

$$\langle \mathcal{E}(\mathbf{n}_1)\mathcal{E}(\mathbf{n}_2)\rangle^{(1)} = \langle \mathcal{E}(\mathbf{n}_1)\mathcal{E}(\mathbf{n}_2)\rangle^{(1)}_{\text{disc}} + \langle \mathcal{E}(\mathbf{n}_1)\mathcal{E}(\mathbf{n}_2)\rangle^{(1)}_{\text{conn}}$$

- Disconnected term measures the phase space

$$\langle \mathcal{E}(\mathbf{n}_1)\mathcal{E}(\mathbf{n}_2)\rangle^{(1)}_{\text{disc}} = -\frac{1}{2}\langle \mathcal{E}(\mathbf{n}_1)\rangle\langle \mathcal{E}(\mathbf{n}_2)\rangle\left(1+(d-1)^2\,\mathbf{n}_1\cdot\mathbf{n}_2\right)$$

- Connected term

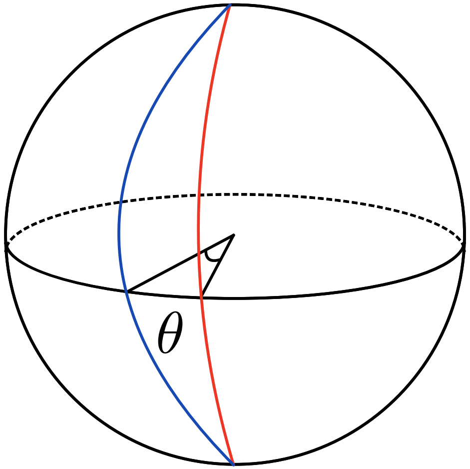
$$\langle \mathcal{E}(\mathbf{n}_1)\mathcal{E}(\mathbf{n}_2)\rangle^{(1)}_{\text{conn}} = \frac{4E^2}{\Delta_Q}\int_{-\pi}^{\pi} dt_1^- \int_{-\pi}^{\pi} dt_2^- \left(\cos\frac{t_1^-}{2}\cos\frac{t_2^-}{2}\right)^d \langle \mathcal{O}_Q| T^{\text{Cyl}}_{--}\, T^{\text{Cyl}}_{--}|\mathcal{O}_Q\rangle_{\text{conn}}$$

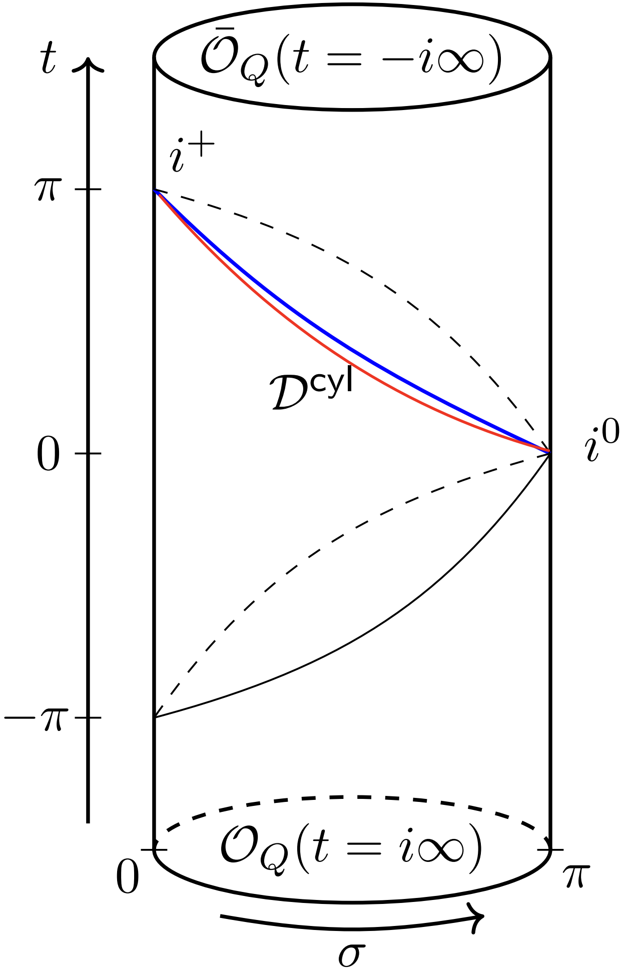
- EFT valid for $\theta = \arccos(\mathbf{n}_1 \cdot \mathbf{n}_2) \gg \frac{1}{\mu}$

but at the endpoints: EFT breaks down in a region of size $1/\mu$

⇓⇓

Local counterterms





Event Shapes of Large Charge States: EFT at the Endpoints

$$\mathcal{E}^{\text{cyl}}(\mathbf{n}_1)\mathcal{E}^{\text{cyl}}(\mathbf{n}_2)|_{\text{UV}} \xrightarrow{\text{RG}} \mathcal{E}^{\text{cyl}}(\mathbf{n}_1)\mathcal{E}^{\text{cyl}}(\mathbf{n}_2)|_{\text{EFT}} + \sum_{y=\text{endpoints}} \mathcal{O}_{ct}(y;\mathbf{n}_1,\mathbf{n}_2),$$

- Structure fixed by conformal invariance, $n_i^\mu = (1, \mathbf{n}_i)$

$$\mathcal{O}_{ct}(y;\mathbf{n}_1,\mathbf{n}_2) = \frac{(\partial\chi)^{2d-4}}{[(n_1\cdot\partial\chi)(n_2\cdot\partial\chi)]^{d-1}} F\left(\frac{(\partial\chi)^2(2n_1\cdot n_2)}{(2n_1\cdot\partial\chi)(2n_2\cdot\partial\chi)}\right) + \dots$$

- Out of EFT corrections

$$\langle\mathcal{O}_Q|\,\mathcal{O}_{ct}(y;\mathbf{n}_1,\mathbf{n}_2)\,|\mathcal{O}_Q\rangle = \mu^{-2} F\left(\frac{1-\mathbf{n}_1\cdot\mathbf{n}_2}{2}\right) \ll \langle\mathcal{O}_Q|\,\mathcal{E}^{\text{cyl}}(\mathbf{n}_1)\mathcal{E}^{\text{cyl}}(\mathbf{n}_2)|_{\text{EFT}}\,|\mathcal{O}_Q\rangle_{\text{Conn}}$$
$$= \mathcal{O}(\mu^d)$$

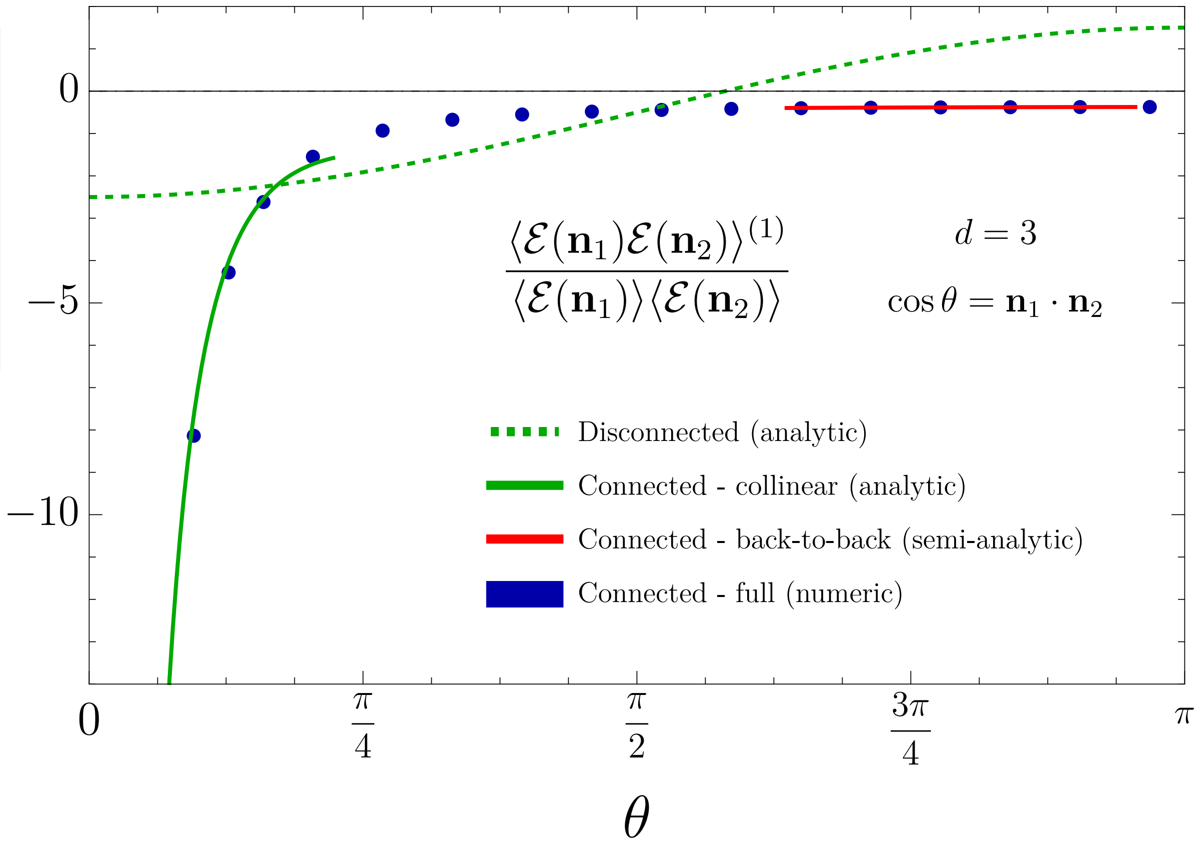
- Collinear limit $1 \gg \theta \gg 1/\mu$

$$\frac{\langle \mathcal{E}(\mathbf{n}_1)\mathcal{E}(\mathbf{n}_2)\rangle^{(1)}}{\langle \mathcal{E}(\mathbf{n}_1)\rangle\langle \mathcal{E}(\mathbf{n}_2)\rangle} \sim -\frac{\#}{\theta^{d-1}}$$

Controlled by the flat space limit of the propagator

$$G_{\pi\pi} \sim \frac{\#}{(\sigma^2 - c_s^2 t^2)^{\frac{d-2}{2}}}$$

Singularity resolved by $i\varepsilon$



Conclusions

- We developed a systematic framework to study event shapes on heavy states in CFTs.
- We computed the next-to-leading correction to event shapes on large charge states.

Future Directions

- Study the event shapes at large charge in a weakly coupled example: match to the OPE regime, match counterterms.
- Compute the subleading correction to event shapes on heavy states with hydrodynamics.

Thank you!